Random deformation of Gaussian fields with an application to Lagrange models for asymmetric ocean waves

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WHY RANDOM DEFORMATION?

In environmental statistics it is common practice to make a transformation (warping) of a random field in order to obtain correlation homogeneity. This paper deals with another type of transformation, where the warping is part of the mechanism that generates the field, and has a physical interpretation.

The statistical distribution of ocean wave slopes is an important parameter in many oceanographic applications, needed for calibration of algorithms in remote sensing of ocean wind and wave fields and many other variables. The distribution is also important for the understanding of physical wave mechanisms and in the safety analysis of marine vessels. A special topic is the front-back asymmetry, the fact that for wind-driven waves, the leeward wave front is steeper than the wave back.

The statistical distribution of wave slopes in general, and front-back asymmetry in particular, has been the subject of systematic empirical studies, since Cox and Munk (1954, 1956). An example of an empirically fitted slope distribution by Cox and Munk, together with a theoretical slope distribution in a first-order Lagrange model, can be seen in Figure 1.

Most theoretical studies of front-back wave asymmetry are based on non-linear differential equation models, and it is difficult to perform a theoretical analysis of their statistical properties. The first-order Lagrange model is a linear model, giving room for explicit statistical analysis. It can reproduce statistical features empirically observed in real ocean waves. The stochastic Lagrange wave model is a realistic alternative to the Gaussian linear model, introduced and studied by Gjøsund (2003), Socquet-Juglard et al. (2004), and Foquet et al. (2006). Theoretical studies have recently been made by Lindgren and Åberg; see references in Lindgren and Lindgren (2011).

We describe a random deformation of a Gaussian field that leads to realistic 3D Lagrange waves with directional spreading, observed at a fixed time over a large area of the sea surface. We present by examples the statistical distributions of slopes observed, with asynchronous area sampling, and synchronous sampling, when observations are taken only at up- or down-crossings of a specified level.

THE 3D STOCHASTIC LAGRANGE MODEL

The first order 3D Lagrange model is a model for the movements of individual water particles on the sea surface. It consists of three correlated Gaussian random fields, $W(t, \boldsymbol{s})$ and $\boldsymbol{\Sigma}(t, \boldsymbol{s}) = (X(t, \boldsymbol{s}), Y(t, \boldsymbol{s})^{\mathsf{T}}$, with time parameter t and space parameter $\boldsymbol{s} = (u, v)$. The parameter $\boldsymbol{s}$ denotes the original horizontal position of a water particle on the surface, and $W(t, \boldsymbol{s}), (X(t, \boldsymbol{s}), Y(t, \boldsymbol{s}))$ are its vertical and horizontal coordinates at time t. The bivariate field $\boldsymbol{\Sigma}$ is thus a random deformation of the field W, taking care of the fact that in real waves, the water particles tend to move in a more or less irregular elliptic patterns, vertically and horizontally.

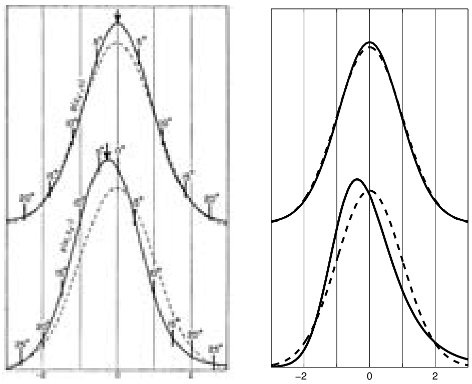

Figure 1: *Left: Illustration from Cox & Munk (1954) of approximate normalized slope density functions in along-wind (lower curves) and cross-wind (upper curves) direction, estimated from sun glitter over an area, together with standard normal densities (dashed). Right: Slope densities calculated in the 3D Lagrange model for Pierson-Moskowitz orbital spectrum with considerable directional spreading. The density has been modified to resemble the Cox and Munk figure, with wave front slopes being positive.*

The fields have mean zero, are stationary in time and homogeneous in space, and they can be expressed as stochastic integrals over wavenumber $\boldsymbol{\kappa} = (\kappa_x, \kappa_y) \in \mathbb{R}^2$, or, alternatively, over wave angular frequency $\omega > 0$ and wave direction $\theta \in (-\pi, \pi]$. Wave number and frequency/direction are related via the dispersion relation, which also includes water depth h. With $\kappa = \sqrt{\kappa_x^2 + \kappa_y^2}$, the dispersion relation is

$$(1) \quad \omega = \omega(\boldsymbol{\kappa}) = \sqrt{g\kappa \tanh \kappa h}, \quad \theta = \arctan_2(\kappa_y, \kappa_x).$$

Here, g is the earth gravitation constant and $\arctan_2$ the four quadrant inverse tangent function.

We denote by τ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ a time difference and a space difference, respectively. The covariance function of the field in space-time is then

$$(2) \quad r^{ww}(\tau, \boldsymbol{\sigma}) = \mathsf{Cov}(W(t, \boldsymbol{s}), W(t+\tau, \boldsymbol{s}+\boldsymbol{\sigma})) = \int_{\omega=0}^{\infty} \int_{\theta=-\pi}^{\pi} \cos(\boldsymbol{\kappa\sigma} - \omega\tau)\, S(\omega, \theta)\, \mathrm{d}\omega\, \mathrm{d}\theta,$$

where $S(\omega, \theta)$, for $\omega > 0$, $-\pi < \theta \le \pi$, is the directional spectrum of the field. We call $S(\omega, \theta)$ the *orbital spectrum*, indicating that it refers to the orbital motions of water particles. A complex representation of the fields are

$$(3) \quad W(t, \boldsymbol{s}) = \int_{(\boldsymbol{\kappa}, \omega) \in D} e^{i(\boldsymbol{\kappa s} - \omega t)}\, \mathrm{d}\zeta^K(\boldsymbol{\kappa}, \omega) = \int_{\omega=-\infty}^{\infty} \int_{\theta=-\pi}^{\pi} e^{i(\boldsymbol{\kappa s} - \omega t)}\, \mathrm{d}\zeta(\omega, \theta).$$

$$(4) \quad \boldsymbol{\Sigma}(t, \boldsymbol{s}) = \begin{pmatrix} X(t, \boldsymbol{s}) \\ Y(t, \boldsymbol{s}) \end{pmatrix} = \boldsymbol{s} + \int_\omega \int_\theta \boldsymbol{H}(\theta, \|\boldsymbol{\kappa}\|)\, e^{i(\boldsymbol{\kappa s} - \omega t)}\, \mathrm{d}\zeta(\omega, \theta).$$

Here $\zeta(\boldsymbol{\kappa}, \omega)$ is a Gaussian complex spectral process with mean 0. The transfer function $\boldsymbol{H}$ is built up with an imaginary part, derived from the hydrodynamic wave equations, and one real part that will account for the deformation. The following form contains a *linkage parameter* α that determines the degree of asymmetry. The choice of the real part is somewhat ad hoc, but is intended to represent the wind influence on the wave dynamics; see further discussion Lindgren and Lindgren (2011) and Lindgren and Aberg (2008).

$$(5) \quad \boldsymbol{H}(\theta, \kappa) = \frac{\alpha}{\omega^2} \cdot \begin{pmatrix} \cos^2(\theta)\,|\cos(\theta)| \\ \cos^2(\theta)\sin(\theta)\,\mathrm{sign}(\cos\theta) \end{pmatrix} + i \frac{\cosh \kappa h}{\sinh \kappa h} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Space and time waves and their derivatives.

The first-order 3D Lagrange model for ocean waves is the tri-variate Gaussian process $(\boldsymbol{\Sigma}(t, \boldsymbol{s}), W(t, \boldsymbol{s}))$, $t \in \mathbb{R}$, $\boldsymbol{s} \in \mathbb{R}^2$. The time dependent Lagrange wave field $L(t, (x, y))$ can be implicitly expressed as

$$(6) \quad L(t, \boldsymbol{\Sigma}(t, \boldsymbol{s})) = W(t, \boldsymbol{s}).$$

This defines $L(t,(x,y))$ uniquely if there are no two $\boldsymbol{s}_1 \neq \boldsymbol{s}_2$ such that $\boldsymbol{\Sigma}(t,\boldsymbol{s}_1) = \boldsymbol{\Sigma}(t,\boldsymbol{s}_2) = (x,y)$. Otherwise, folding occurs, and the Lagrange wave has several branches. By keeping either time or space coordinates fixed, $t = t_0$, and $(x,y) = (x_0,y_0)$, respectively, one obtains two types of wave observations, recorded in empirical studies, space waves, and time waves, respectively.

The wave slopes in the two models can be expressed in terms of the partial derivatives, found by direct differentiation of (6):

$$(7)\quad \begin{pmatrix} W_u \\ W_v \end{pmatrix} = \begin{pmatrix} X_u\, Y_u \\ X_v\, Y_v \end{pmatrix} \begin{pmatrix} L_x \\ L_y \end{pmatrix}, \qquad W_t = L_t + \begin{pmatrix} X_t\, Y_t \end{pmatrix} \begin{pmatrix} L_x \\ L_y \end{pmatrix},$$

where L_t, L_x, L_y denote the local partial derivatives, on the branch determined by the reference coordinates.

The Lagrange space wave is what is seen in a photo or radar image of the sea surface, like in the study by Cox and Munk (1954). The statistical distribution of space wave characteristics, such as crest height, wave length, etc, are to be interpreted in a frequentistic way as what one can empirically observe from observations of an infinitely extended, statistically homogeneous, section of the ocean, with observations either by *asynchronous sampling* at a fixed grid in space or by *synchronous sampling* at locations where level crossings occur.

We will now show some examples of slope distributions, obtained by asynchronous or synchronous sampling. The theory behind the distributions is presented in Lindgren and Lindgren (2011).

EXAMPLES

The purpose of this example is to show how the degree and direction of the spreading affects the front-back asymmetry of the Lagrange space waves with different degree of linkage in model (5). To clearly see the effects, we have chosen a moderate water depth, $h = 32$ m.

We will illustrate the theory on a model with Pierson-Moskowitz (PM) orbital frequency spectrum, i.e. the spectrum of the W-field, with un-directional, one-sided spectral density

$$S(\omega) = \frac{5H_s^2}{\omega_p(\omega/\omega_p)^5}\, e^{-\frac{5}{4}(\omega/\omega_p)^{-4}}, \quad 0 \le \omega \le \omega_c,$$

where $H_s = 4\sqrt{\mathsf{V}(W(t,u))}$ is the *significant wave height* in the W-process, and ω_p is the *peak frequency*, at which the spectral density has its maximum. The *peak period* is defined as $T_p = 2\pi/\omega_p$. We use fixed values, $H_s = 7$ m and $T_p = 11$ s, for significant wave height and peak period, and assume a finite cut off frequency $\omega_c = 2.5$ rad/s to avoid small but high frequency wave components.

The unidirectional space waves, without spreading, were studied by Lindgren and Aberg (2008), and the results from that paper will be used for comparison. Here, the directional spreading is taken as frequency independent and defined by the $\cos 2\theta$-function, so

$$(8)\quad S(\omega,\theta) = c(m)\, S(\omega) \cos^{2m}\left(\frac{\theta-\theta_0}{2}\right),$$

with different values for the spreading parameter m. An isotropic wave field corresponds to $m = 0$, while $m = \infty$ resembles unidirectional waves. We use $m = 0, 2, 5, 10, 20, 120$, in this example. All simulations and computations are made in the MATLAB toolbox WAFO; see [8].

Asynchronous sampling

Figure 3 shows densities of asynchronous slopes in x- and y-direction ("along-wind" and "cross-wind") for different degrees of linkage and spreading in model (5). These distributions resemble the

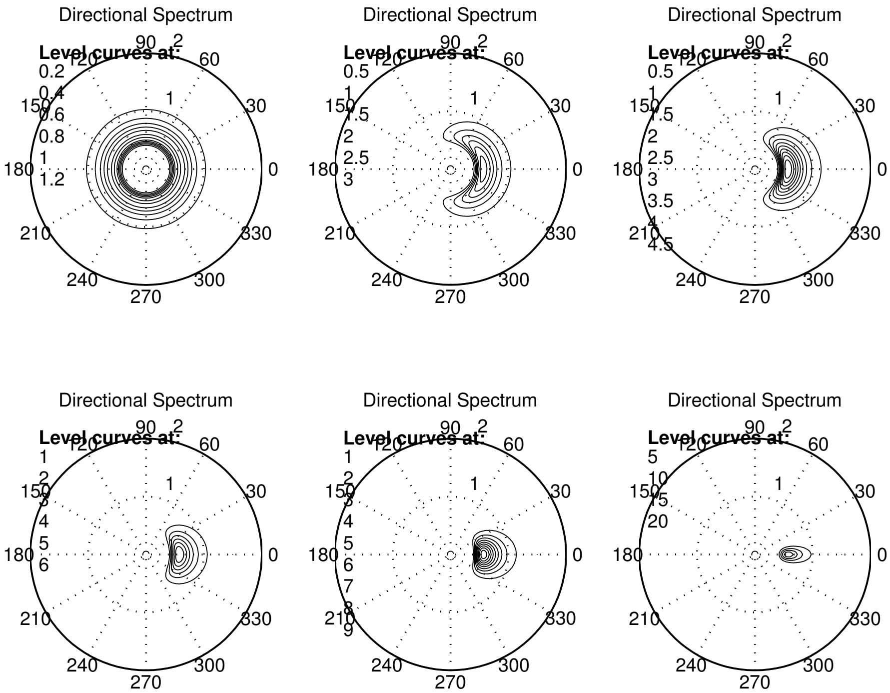

Figure 2: Directional Pierson-Moskowitz (PM) spectra for the $W(t,\boldsymbol{s})$-field, $S(\omega,\theta) = c(m)\,S(\omega)\cos^{2m}(2\theta)$; $m = 0, 2, 5, 10, 20, 120$, from top-left to bottom-right.

distributions obtained by Cox and Munk (1954), shown in Figure 1. Note that the densities in the right panel in Figure 1 are turned backwards to be compatible with the original Cox and Munk figure.

The figures reveal some interesting features of the linked Lagrange model. The asymmetry between the positive and negative parts of the asynchronous slope distribution is only present for the more extreme part of the two distributions. For slopes smaller than the median there is little asymmetry; this is verified from the complete data. For slopes larger in absolute value than the median in the two distributions, the asymmetry is large.

The effect of directional spreading is clear from the figure. In the more concentrated spectrum, with $m = 20$, all the mentioned effects are more pronounced than for $m = 5$.

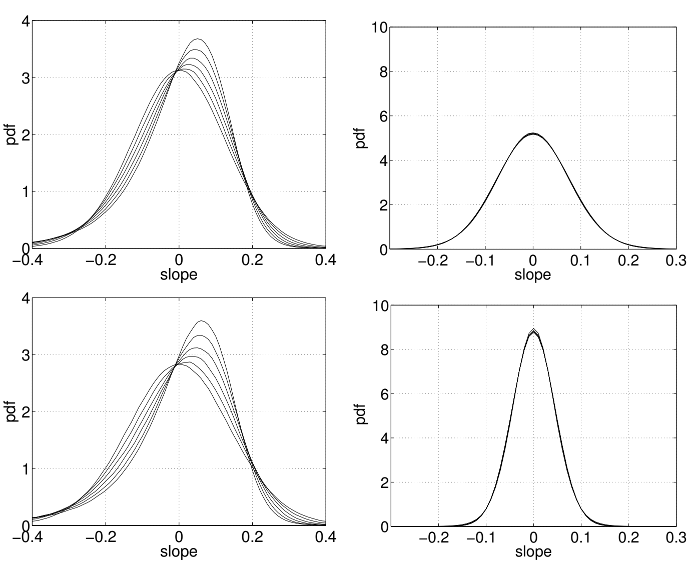

Figure 3: Asynchronous slope PDF in x- (left) and y-directions (right). Orbital spectrum is PM with water depth $h = 32$ m. Directional spreading according to Figure 2 with $m = 5$ (upper panel) and $m = 20$ (lower panel). Linkage parameters are $\alpha = 0, 0.4, 0.8, 1.2, 1.6, 2.0$ (most skewed).

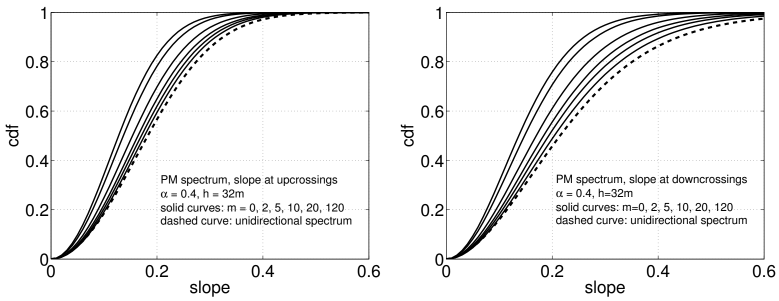

Figure 4: Cumulative distribution functions (CDF) for (absolute) space slopes at up- and downcrossings of the level $w_0 = \sigma = H_s/4$ along the main wave direction $\theta_0 = 0$. PM-spectra with different m. The linkage parameter in model (5) is $\alpha = 0.4$.

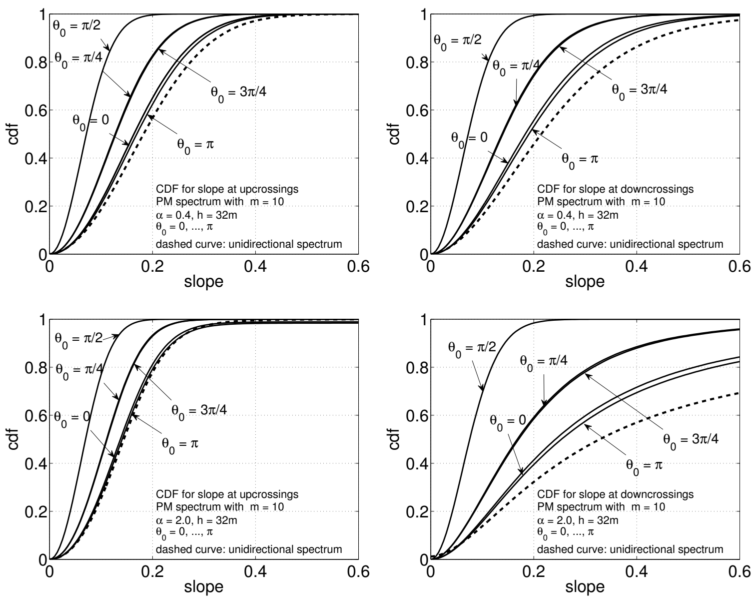

Figure 5: Mean wave direction's influence on the CDF for (absolute) space slopes at up- (left) and downcrossings (right) along the x-direction of the level $w_0 = \sigma = H_s/4$. Spreading $m = 10$, and linkage parameter in model (5) is $\alpha = 0.4$ (upper panel) and $\alpha = 2.0$ (lower panel).

Synchronous sampling – Slopes at up- and downcrossings

This example illustrates the synchronous sampling of slopes observed at up- and downcrossings of specified levels. The distribution function for slopes in the space waves, observed along the positive x-axis, i.e. in the main wave direction, is computed as described in Lindgren and Lindgren (2011).

Figure 4 shows the cumulative distribution functions for the (absolute values of the) slopes at upcrossings and downcrossings of the level $w_0 = \sigma = H_s/4$ for the six degrees of directional spreading in Figure 2. For comparison, the distribution for the unidirectional case is also plotted (dashed curve). Obviously, the wave steepness decreases with increasing directional spreading, a fact that agrees with many other theoretical and empirical wave studies, as well as with the results for asynchronous sampling. Also the front-back asymmetry decreases with increasing spreading, as can be expected.

As a final example, Figure 5 shows how the asymmetry depends on the mean wave direction in relation to the observation axis for the space waves. The spectrum is the directional PM-spectrum, (8), with $m = 10$, and main wave direction $\theta_0 = 0, \pi/4, \pi/2, 3\pi/4, \pi$. The linkage parameter in (5) is taken as $\alpha = 0.4$ as in Figure 4, and as $\alpha = 2$, for a more extreme case.

As see from the figure, the linkage has almost no effect on the up- or downcrossing slopes when the linkage (wind) is parallel to the wave crests, $\theta_0 = \pi/2$. The slopes at upcrossings are slightly smaller than at the downcrossings but the difference is small. Wind against the main wave direction gives steeper waves than wind along the wave direction, and the same holds for the intermediate cases, $\theta_0 = \pi/4$ and $\theta_0 = 3\pi/4$.

Acknowledgment

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ABSTRACT

In the stochastic Lagrange model for ocean waves the vertical and horizontal movements of surface water particles are modeled as correlated Gaussian processes. In this paper we investigate the statistical properties of wave characteristics related to wave asymmetry in the 3D Lagrange model. We present a modification of the original Lagrange model that can produce front-back asymmetry both of the space waves, i.e. observation of the sea surface at a fixed time, and of the time waves, observed at a fixed measuring station. The results, which are based on a multivariate form of Rice's formula for the expected number of level crossings, are given in the form of the cumulative distribution functions for the slopes observed either by asynchronous sampling in space, or at synchronous sampling at upcrossings and downcrossings, respectively, of a specified fixed level. The theory is illustrated in a numerical section, showing how the degree of wave asymmetry depends on the directional spectral spreading and on the mean wave direction. It is seen that the asymmetry is more accentuated for high waves, a fact that may be of importance in safety analysis of capsizing risk.

Keywords: Crossing theory, directional spreading, front-back asymmetry, Gaussian process, Palm distribution, Rice formula, slope asymmetry, wave steepness.