Statistical Estimation of the Age of the Universe

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Abstract

The Hubble constant enters big bang cosmology by quantifying the expansion rate of the universe. It is shown that the standard technique for estimation of Hubble's constant is statistically inconsistent and results in a systematically too low value. An alternative, consistent estimator of Hubble's constant is presented resulting in a change in estimate of 1.2%. The new estimate implies that the universe is 170 million years younger than previously thought.

Introduction

In the 1920s Edwin Hubble studied Cepheid variable stars in spiral nebulae (what is now called galaxies). A Cepheid variable is a star that has distinct brightness periodicity which is used to establish its distance from the Earth. Surprisingly, the observed Cepheids were found to be at distances which placed them well outside the Milky Way.

Hubble found that each Cepheid variable in interstellar space had a Doppler shift corresponding to an observable relative velocity to the Earth. Although not in itself surprising because stars conceivably have some velocities relative to the Earth, it was surprising that the stars all had Doppler red shifts corresponding to them moving away from Earth. Further, it turned out empirically, that the velocity by which the stars where moving away was higher as the distance to the stars got larger. The relationship between velocity and distance is expressed by Hubble’s Law.

Hubble’s law states that the velocity at which a galaxy is receding from the Earth is proportional to its distance from the Earth. The law is considered the first observational basis for the expanding space paradigm. Hubble's law, in conjunction with the microwave background radiation, are the most often cited pieces of evidence in support of the Big Bang model.

Measuring an accurate value of the Hubble constant (H_0) was one of three “Key Projects” of the NASA/ESA Hubble Space Telescope (HST) project. The constant plays an important part in cosmology and astrophysics. Notably, its inverse, H_0^{-1} sets the age of the universe. It is also important when calculating the critical density, the total energy density and the observable size of the universe.

Estimation of Hubble’s constant comes from different sources including the Wilkinson Microwave Anisotropy Probe (WMAP) which measures differences in the temperature of the Big Bang’s remnant radiant heat, gravitational lensing, and, as was originally done by Hubble, by relating distances and velocities of different types of stars, including Cepheids, but also exploding stars (supernovae) which can be seen at a great distance.

Accurate estimation of the Hubble constant comes from Type Ia supernovae. These supernovae arise from the thermonuclear explosions of carbon-oxygen white dwarfs that have accreted mass off the companion star and gradually grown to the Chandrasekhar mass (1.38 times the solar mass). The supernovae are known as “standard candles” because they produce consistent peak luminosities. This is exploited to measure their distance from Earth because the magnitude of the observed luminosities depend primarily on the distance from the supernovae.

The Type Ia Supernovae Data

The data presented here are based on 36 type Ia Supernovae, and are published in the final results from the Hubble Space Telescope Key Project to measure the Hubble constant [FRE 01]. Figure 1 shows the data.

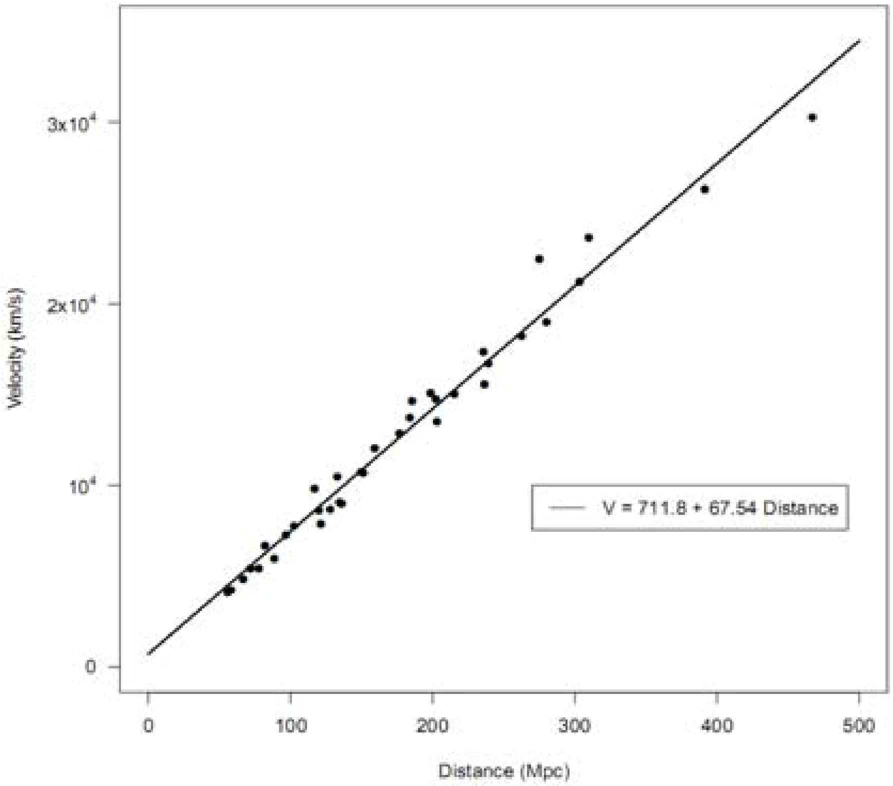

Figure 1. *Velocity versus distance for 36 Type Ia supernovae. The line is the ordinary least squares estimator.*

In Figure 1, the usual SI unit for distance (meter), is replaced by parsec which is customary in astronomy. A parsec (parallax of one arcsecond; symbol: pc) is just under 31 trillion kilometres, or about 3.26 light-years. It relates the distance to the associated angle at which a star is measured at different time points – e.g. summer and winter. A parsec is the distance from the Sun to an astronomical object which has a parallax angle of one arcsecond. The closest star (not including our own sun) is Proxima Centauri with a parallax of 0.77 arcseconds; it is thus 1.3 pc (4.2 light years) from Earth. At present, we can measure distances to about 1000 pc with reasonable precision. Our neighbouring galaxy, The Andromeda Galaxy, is 0.77 Mpc away from the Earth.

To measure longer distances than 1000 pc other methods are used, and the parallax method works as the fundamental calibration step for distance determination in astrophysics. The calibration step to enable measurement of very distant stars is complicated, but relies on using well established techniques, e.g. the parallax method, in conjunction with a new technique, e.g. Cepheids variable stars with a distinct brightness periodicity. This establishes the new technique as a measuring device further out in space which is then, in turn calibrated against yet another technique, notably type Ia Supernovae. The successive calibrating steps are sometimes called the cosmic distance ladder or the Extragalactic Distance Scale.

The relationship between velocity (as measured by the redshift in spectral lines from the star) and the distance (calculated using the distance ladder) is seen in Figure 1. The apparent linear relationship was first noted by Hubble and is now called Hubble's law. It represents the last step of the distance ladder since it is the primary means for estimating the distances of quasars and distant galaxies in which individual distance indicators, e.g. supernovae, cannot be seen.

Initial Analysis

An initial linear regression analysis, using the ordinary least squares method) reveals a surprising finding shown in Table 1.

Table 1

	Estimate	Std. Error	*t* value	$\Pr(>\|t\|)$
Intercept	711.8	347.4	2.0	0.048
Slope	67.5	1.7	39.1	<0.001

The regression line is shown in Figure 1, and what is noticeable is that the intercept – interpreted as the velocity away from Earth at the position of the Earth – is significantly different from zero. The speed of 712 km/s is substantial (the escape velocity from earth is 11 km/s).

Actually, the above analysis does not correspond to Hubble's law which stated formally is

$$V_i = H_0 D_i,\ i = 1, \ldots, 36$$

where D_i is the distance and V_i the velocity. H_0 is Hubble's constant.

Hubble's law avoids the possible inconsistency of an intercept which is not zero by setting it to zero (omitting it in the model). Estimating Hubble's constant in this model yields an ordinary least squares (OLS) estimate of 70.67 km s^{-1} Mpc^{-1} with a standard error of 0.84 km s^{-1} Mpc^{-1}. These numbers are reported in Table 12 in the final results from the Hubble Space Telescope Key Project to measure the Hubble constant, see (1).

Notice that the slope estimator in Table 1, i.e. when including also the intercept, is somewhat lower and with a higher standard error. The difference between the two estimates may be the result of measurement error in the distance measurement, resulting in what is known in statistics as regression towards the mean.

That there is a problem with measurement error is beyond discussion, since measuring intergalactic distances in astronomy is notoriously difficult and error prone.

Performing the reverse regression analysis, i.e. regression instead distance on velocity, results in a different estimate of Hubble's constant: 1/0.0140815= 71.02 km s^{-1} Mpc^{-1}. As the errors in the velocity measurements are of a smaller magnitude, this result is probably a better estimate.

In the next section, a new estimator is presented which allows for measurement error in both the velocity and the distance measurements.

Corrected Analysis

In the following, we set up a model framework and investigate the properties of the resulting slope estimators. We denote by D_i and V_i, respectively, the true distance from Earth to the *i*th supernova, and the velocity of the supernova relative to Earth. Further X_i and Y_i represent the corresponding measured distance and measured velocity, respectively. The relation between *D* and *X* is $X_i = D_i + \delta_i$ where δ_i is the measurement error on the distance. Similarly, the observed velocity can be written $Y_i = V_i + \varepsilon_i$. From Hubble's law, it follows that $Y_i = H_0 D_i + \varepsilon_i,\ i = 1, \ldots, 36,$ where ε_i where ε_i is the measurement error on the velocity. For convenience, the measurement errors are assumed independent and identically distributed mean zero Gaussian error terms with a variance σ^2. It follows that Hubble's constant satisfies $H_0 = E\{YD\} \big/ E\{D^2\}$ which could be estimated by its natural empirical counterpart – the ordinary least squares estimator, $\sum Y_i D_i \big/ \sum D_i^2$. As *D* is not observed (only *X* is), standard practice has been to replace *X* for *D*. In fact, this is the ordinary least-squares estimator for the model $Y_i = \tilde{H}_0 X_i + \varepsilon_i$.

Expressing $\tilde{H}_0$ in terms of the true Hubble constant (H_0) yields

$$\tilde{H}_0 = \frac{E\{YX\}}{E\{X^2\}} = \frac{E\{YD\} + E\{Y\}E\{\delta\}}{E\{D^2\} + E\{\delta^2\} + 2E\{D\}E\{\delta\}} = \frac{E\{YD\}}{E\{D^2\} + V\{\delta\}} < H_0 \ .$$

The last in-equality is a result of the error on the distance measurements being non-negligible. This shows that the usual regression analysis in which the observed distance measurements *X* are substituted for the true (but unknown) distances *D* results in a deflated estimate of Hubble's constant.

To alleviate this problem, a statistically consistent estimator is proposed. The Hubble constant satisfies $H_0 = E\{Y\}/E\{D\}$ and therefore also $H_0 = E\{Y\}/E\{X\}$. The natural empirical counterpart is therefore $\sum Y_i / \sum X_i$. Note that this estimator has the nice property that doing the reverse regression analysis, i.e. regressing distance on velocity, will result in exactly the same estimate of Hubble's constant.

Both numerator and denominator are by the central limit theorem asymptotically Gaussian. With Gaussian numerator and denominator, the exact probability density function of the ratio is $f(z) = \int |u| \phi(zu, u)\, du$, where the integral is over the whole real line [HIN 69]. The function ϕ is the joint Gaussian distribution density of $(\sum Y_i, \sum X_i)$. From this expression we derive a confidence interval for Hubble's constant. The estimator has superior properties assuming only that the measurement error on the distance has zero mean and is independent of the velocity measurement. In contrast, for OLS to yield a consistent estimate, zero measurement error would be required.

The data are shown along with the two estimators in Figure 2.

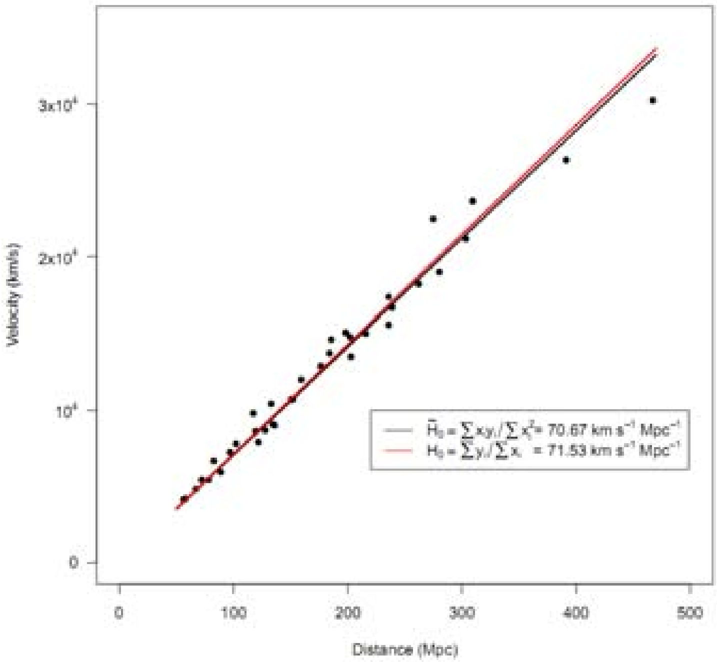

Figure 2. Hubble diagram of observed velocity versus observed distance for 36 Type Ia supernovae. The black line is the traditionally used, and as shown inconsistent, ordinary least-squares (OLS) slope estimator of the Hubble Constant. The red line is the consistent estimator of Hubble's constant. The difference between the two estimates is 1.2%.

The OLS estimator gives 70.67 km $s^{-1}Mpc^{-1}$ with a standard error of 0.84 km $s^{-1}Mpc^{-1}$. A 95% confidence interval is (68.97, 72.37) km $s^{-1}Mpc^{-1}$. The consistent estimator gives 71.53 with a confidence interval (69.74, 73.57) km $s^{-1}Mpc^{-1}$. The price for obtaining consistency is seen from Figure 3 to be a slighter wider confidence interval.

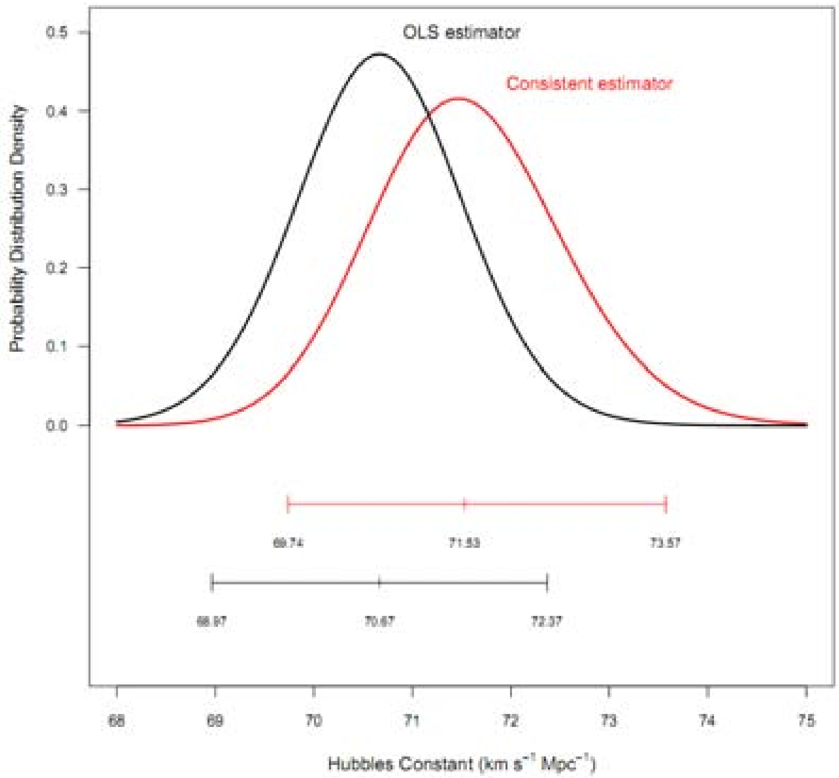

Figure 3. Probability distribution densities of the ordinary least squares (OLS) estimator and the consistent estimator.

Calculating the age of the universe

Hubble's law, although simple to formulate mathematically, has mind bottling consequences. It forced a change in the existing notion of space as something static and everlasting. Also the notion of a Big Bang is given a much more sophisticated interpretation than one might have thought. Rather than being an explosion in an existing space, it is an expansion of space itself. An often used analogy used to illustrate the expanding space is that of an elastic band being pulled. Points that are initially close to each other will, when the band is pulled, move apart slower that points that were initially farther apart. The same type of expansion is envisaged in space

A simple consequence of Hubble's law, is that the age of the universe can be calculated. Consider a star moving towards Earth with a velocity *V* and reaching Earth after time *t*. From Hubble's law the velocity for a star at a distance *A* from Earth is $V = H_0 \times A$. The distance that the star travel during the time *t* is $V \times t$ which is also A. These two results combined yields $V = H_0 \times A = H_0 \times V \times t$ which implies that $t = H_0^{-1}$. This result does not depend on the distance away from Earth of the star. So, $t = H_0^{-1}$ ago all stars were together in on point – and Earth was there as well. Then came the Big Bang.

Note that the time unit of H_0 is a little unusual, so to get to an age in years we need a conversion factor. Using that 1Mpc is 1000000 parsec, that 1 parsec is $360 \times 60 \times 60 / 2 / \Pi$ astronomical unit (AU) and that 1 AU is 149597870.691 km, the conversion factor becomes $k = 0.9777655 \times 10^{12}$. So $t = k \times H_0^{-1}$ is the age of the universe in seconds which is readily converted into years.

Using the OLS estimate of Hubble's constant, the age becomes 13.85 billion years. With the consistent estimate, the age becomes 13.67 billion years (95% CI: 13.51 – 14.02 billion years). The difference of 1.2% amount to 170 million years.

Discussion

The approach presented here, published in a short form in [PET 10], may serve as a framework for future estimation of Hubble's constant based on the still mounting data from different sources including for example Type II Supernovae and different clusters. Only Type Ia supernovae are included here because the data for these objects as well as the OLS estimate of Hubble's constant, is described in detail in [FRE 01]. Furthermore, with their large distances from Earth they provide excellent statistical estimation of Hubble's constant.

The inconsistency of the OLS estimator, and the magnitude of the deflation of the estimate, is a result of the measurement error on the distances, properties that remain unchanged no matter how many more observations are gathered. In contrast, the new approach results in Hubble's constant being estimated with better and better precision as more and more velocity/distance measurements become available. Note, that systematic errors pertaining to for example calibrations of measurement devices, data corrections etc. are not discussed here. These sources of error remain and are unaffected by these considerations.

The techniques used for estimating Hubble's constant in practice are of course more complicated than the OLS estimator presented here. Multiple analysis techniques and multiple estimators are used including Bayesian analyses. The point stressed here is that these techniques build on the simple OLS technique, and that all the techniques use the same approach to the measurement error problem, and are all biased! The bias problem is just much simpler to recognize when peeling of all the details like combining data sources using a distance ladder, and including prior distributions in a Bayesian analysis.

We have not discussed estimation in a more realistic setting in which many sources of data are combined. Three extensions of what has been presented here are envisaged: One, a weighted regression estimator which puts weight on distance and velocity measurements according to the precision by which they are measured. Two, a Maximum Likelihood Estimator based on a full statistical model of the bivariate distribution of the velocity and distance measurements. This latter approach, although fully efficient, relies on distributional assumptions which have to be addressed. Three standard calibration techniques of the ordinary least squares estimator. Also, a standard Bayesian approach may be taken allowing one to include prior knowledge. Otherwise the technical aspects of this approach are similar to the others mentioned.

A more theoretical description of the distribution of the proposed estimator including discussion of sample size dependent bias correction and bias-variance trade-offs in the different estimators

The analysis demonstrates that the estimation of Hubble's constant by ordinary least squares results in an estimate which is statistically inconsistent and systematically underestimates the true value. The underestimation is a statistical artefact resulting from not taking the measurement error on the distance measurements into account in the analysis. Importantly, the inconsistency is an intrinsic property of the method. It does not disappear, or even diminish, in the presence of more velocity/distance measurements. Only in the unlikely scenario, in which all measurement errors on the distance was eliminated, could this problem be disregarded. Instead, another estimator of Hubble's constant is presented and it is shown to be consistent. The final result is an improvement in the estimate by 1.2% corresponding to one standard error.

References

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