The role of the real representation- in quaternion distribution theory

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1 Introduction

The real representation of quaternion distributions made their debut in the statistical literature in papers by Kabe ([7], [8], [9]). Rautenbach [13] and Rautenbach and Roux ([14], [15]) also utilised the representation theory in their papers, and has been revived more recently by Teng and Fang [16].

In Section 2 a collection of fundamental mathematical results are presented for use in later sections. Section 3 is first devoted to a review on the derivation of the p-variate quaternion normal distribution, using the representation theory, after which the matrix-variate quaternion normal distribution is derived by generalising this approach. We conclude in Section 4 by illustrating how this approach may be applied, for example how the quaternion Wishart distribution relates to its real counterpart.

The real counterpart approach should not merely be seen as a collection of corollaries of the work done in real normed division algebras (see for example [2],[3], [4], [10] and [11]), but rather as a necessity for a thorough grip on the hypercomplex distribution theory.

2 Mathematical Preliminaries

Let $\mathbb{R}$ denote the field of real numbers, and $\mathbb{Q}$ the quaternion (Hamiltonion) division algebra over $\mathbb{R}$, respectively. Hence, every $z \in \mathbb{Q}$ can be expressed as

$$z = x_1 + ix_2 + jx_3 + kx_4,$$

where i, j, and k satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k,\ jk = -kj = i, ki = -ik = j$$

and where $x_1, x_2, x_3 \in \mathbb{R}$. The conjugate of a quaternion element is defined in a similar fashion to that of a complex number, and is given by:

$$\bar{z} = x_1 - ix_2 - jx_3 - kx_4$$

Now let $M_{n\times p}(\mathbb{R})$ and $M_{n\times p}(\mathbb{Q})$ denote the set of all $n \times p$ matrices over $\mathbb{R}$ and $\mathbb{Q}$, respectively. In the case of square matrices, say $p \times p$, this will be indecated by $M_p(\mathbb{R})$ and $M_p(\mathbb{Q})$ instead. Similar to the scalar form above, any $\mathbf{Z} \in M_{n\times p}(\mathbb{Q})$ may be rewritten as:

$$\mathbf{Z} = [z_{ij}]_{n\times p} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4,$$

where $z_{ij} \in \mathbb{Q}$, and $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{X}_4 \in M_{n\times p}(\mathbb{R})$. $\mathbf{X}_1$ is the real part of $\mathbf{Z}$, and will be denoted by $\mathrm{Re}\,\mathbf{Z}$. By setting $n = 1$, this reduces to the vector form in an obvious way.

We will denote the transpose of a matrix $\underset{n\times p}{\mathbf{Z}}$ as $\mathbf{Z}' = \underset{p\times n}{\mathbf{Z}}$. The conjugate transpose of $\mathbf{Z}$ is therefore given by

$$\bar{\mathbf{Z}}' = \left[\bar{z}'_{ij}\right]_{p\times n} = \mathbf{X}'_1 - i\mathbf{X}'_2 - j\mathbf{X}'_3 - k\mathbf{X}'_4$$

and we say $\mathbf{Z}$ is Hermitian if $\bar{\mathbf{Z}}' = \mathbf{Z}$.

The vec operator is frequently used in expressions involving matrices of quaternions, see for instance Li and Xue [11], and is defined as

$$\text{vec}\mathbf{Z} = \left[\underline{Z}'_1, \ldots, \underline{Z}'_p\right]' \in M_{np\times 1}(\mathbb{Q}),$$

where $\underline{Z}_\alpha \in M_{n\times 1}(\mathbb{Q}), \alpha = 1, \ldots, p$ are the columns of $\mathbf{Z}$.

We will make use of the representation theory throughout this paper, and although quaternions may be represented by real matrices in various ways, see for instance Teng and Fang [16], we will use the representation employed by Kabe (for instance [7] and [9]) and Rautenbach [13]. Specifically suppose that $z = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{Q}$ may be represented by $\mathbf{z}_0 \in M_4(\mathbb{R})$, as

$$\underset{4\times 4}{\mathbf{z}_0} = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}.$$

Now, if $\underset{n\times p}{\mathbf{Z}} \in M_{n\times p}(\mathbb{Q})$, i.e. the case where we have matrices with quaternion elements (or vectors by setting $n = 1$), we have $\underset{n\times p}{\mathbf{Z}} = [z_{st}]$, where $z_{st} = x_{1st} + ix_{2st} + jx_{3st} + kx_{4st} \in \mathbb{Q}, s = 1, \ldots, n,$ and $t = 1, \ldots, p$. By an elementwise generalisation of our representation of the scalar, to the matrix (or vector) form, we have

$$\underset{4\times 4}{\mathbf{z}_{0st}} = \begin{bmatrix} x_{1st} & -x_{2st} & -x_{3st} & -x_{4st} \\ x_{2st} & x_{1st} & -x_{4st} & x_{3st} \\ x_{3st} & x_{4st} & x_{1st} & -x_{2st} \\ x_{4st} & -x_{3st} & x_{2st} & x_{1st} \end{bmatrix}$$

in other words, $\underset{n\times p}{\mathbf{Z}}$ may be represented with the real matrix $\underset{4n\times 4p}{\mathbf{Z}_0}$ as: $\underset{4n\times 4p}{\mathbf{Z}_0} = [\mathbf{z}_{0st}]$.

Moreover, if we define the mapping

$$f\left(\underset{4p\times 4p}{\mathbf{R}}\right) = \underset{p\times p}{\mathbf{Q}} \ \forall \mathbf{R} \in M_{4p}(\mathbb{R}), \mathbf{Q} \in \mathbb{Q},$$

it was shown in Rautenbach [13] that f is a faithful representation. When $\underset{4p\times 4p}{\mathbf{R}} \in M_{4p}(\mathbb{R})$ and $f(\mathbf{R}) = \mathbf{Q} \in M_p(\mathbb{Q})$ then it will be indicated as $\mathbf{R} \simeq \mathbf{Q}$.

If $\mathbf{Q} \in M_p(\mathbb{Q})$ is a Hermitian positive definite matrix, then its eigenvalues $\lambda_s, s = 1, \ldots, p$ are real and positive and there exists a Hermitian positive matrix, written as $\mathbf{Q}^{\frac{1}{2}}$ such that $\mathbf{Q} = \mathbf{Q}^{\frac{1}{2}}\mathbf{Q}^{\frac{1}{2}}$, see Teng and Fang [16]. For more technical results and a review on quaternion matrix algebra, see Zhang [17].

The trace operator is frequently used in the symplification of expressions, and although the multiplication of quaternions are noncommutative, we may go about it as follows, see Zhang [17]. Let Re tr($\mathbf{A}$) =tr(Re $\mathbf{A}$) for $\mathbf{A} \in M_p(\mathbb{Q})$, we have

$$(1) \quad \begin{aligned} \operatorname{Re}\operatorname{tr}(\mathbf{A}) &= \tfrac{1}{2}\operatorname{tr}(\mathbf{A} + \bar{\mathbf{A}}') \\ \operatorname{Re}\operatorname{tr}(\mathbf{AB}) &= \operatorname{Re}\operatorname{tr}(\mathbf{BA}) \quad \forall \mathbf{A}, \mathbf{B} \in M_p(\mathbb{Q}) \end{aligned}$$

Moreover, if $\mathbf{A} = \bar{\mathbf{A}}' \in M_p(\mathbb{Q})$, i.e. a Hermitian matrix, then this becomes

$$\operatorname{Re}\operatorname{tr}(\mathbf{A}) \quad = \operatorname{tr}(\mathbf{A}) = \sum_{\alpha=1}^{p} \lambda_\alpha,$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $\mathbf{A}$.

We now define some concepts specifically pertaining to the development of the quaternion distribution theory, and show how they interact with their real associated counterparts.

Let $\underset{n\times p}{\mathbf{Z}} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4$ be a quaternion probability matrix. The associated real probability matrix is given by

$$\underset{n\times 4p}{\mathbf{Z}_0} = \left[\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3', \mathbf{X}_4'\right].$$

Definition 1 Let $\underset{n\times p}{\mathbf{Z}} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4$ be a quaternion probability matrix. The characteristic function of $\mathbf{Z}$ is defined as

$$(2) \qquad \phi_{\mathbf{Z}}(\mathbf{T}) \quad = E\left[\exp \tfrac{\iota}{2} tr\left(\bar{\mathbf{Z}}'\mathbf{T} + \bar{\mathbf{T}}'\mathbf{Z}\right)\right],$$

where $\underset{n\times p}{\mathbf{T}} = \mathbf{T}_1 + i\mathbf{T}_2 + j\mathbf{T}_3 + k\mathbf{T}_4$ is a quaternion matrix and ι is the usual imaginary complex unit.

If we let $\mathbf{V} = \bar{\mathbf{Z}}'\mathbf{T} + \bar{\mathbf{T}}'\mathbf{Z}$, say, then it follows that $\bar{\mathbf{V}}' = \bar{\mathbf{T}}'\mathbf{Z} + \bar{\mathbf{Z}}'\mathbf{T} = \mathbf{V}$ implying that $\mathbf{V}$ is Hermitian and hence, from (1), the characteristic function may be written as

$$(3) \qquad \phi_{\mathbf{Z}}(\mathbf{T}) \quad = E\left[\exp \tfrac{\iota}{2} \operatorname{Re}\operatorname{tr}\left(\bar{\mathbf{Z}}'\mathbf{T} + \bar{\mathbf{T}}'\mathbf{Z}\right)\right] \quad = E\left[\exp \iota \operatorname{Re}\operatorname{tr}\left(\bar{\mathbf{Z}}'\mathbf{T}\right)\right]$$

In the case of a quaternion probability vector, $\underset{p\times 1}{\underline{Z}} = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4$, (2) reduces to

$$(4) \qquad \phi_{\underline{Z}}(\underline{t}) = E\left[\exp \frac{\iota}{2}\left(\bar{\underline{Z}}'\underline{t} + \bar{\underline{t}}'\underline{Z}\right)\right],$$

where $\underset{p\times 1}{\underline{t}} = \underline{t}_1 + i\underline{t}_2 + j\underline{t}_3 + k\underline{t}_4$ is a quaternion vector and ι is the usual imaginary complex unit. An expression for the characteristic function of $\underset{p\times 1}{\underline{Z}}$ in terms of Re tr can be derived in a similar fashion as in (3) and is given by

$$\phi_{\underline{Z}}(\underline{t}) = E\left[\exp \iota \operatorname{Re}\operatorname{tr}\left(\bar{\underline{Z}}'\underline{t}\right)\right].$$

The nature of the problem will determine the form of the characteristic function used in its derivation.

It now follows that

$$\phi_{\mathbf{Z}}(\mathbf{T}) = E\left[\exp \iota \operatorname{Re}\operatorname{tr}\left(\mathbf{X}_1'\mathbf{T}_1 + \mathbf{X}_2'\mathbf{T}_2 + \mathbf{X}_3'\mathbf{T}_3 + \mathbf{X}_4'\mathbf{T}_4\right)\right]$$

such that

$$(5) \qquad \phi_{\mathbf{Z}}(\mathbf{T}) = \phi_{\mathbf{Z}_0}(\mathbf{T}_0).$$

$\phi_{\mathbf{Z}_0}(\mathbf{T}_0)$ is the characteristic function of the associated real probability matrix, $\underset{n\times 4p}{\mathbf{Z}_0} = [\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3', \mathbf{X}_4']$ and further is $\underset{n\times 4p}{\mathbf{T}_0} = [\mathbf{T}_1', \mathbf{T}_2', \mathbf{T}_3', \mathbf{T}_4']$ a real matrix. It is therefore clear that the characteristic function of a quaternion probability matrix is equivalent to the characteristic function of a $n \times 4p$-variate real probability matrix. A similar result holds for the special case in which $n = 1$, yielding a quaternion probability vector.

3 The quaternion normal distribution

The p-variate quaternion normal distribution forms the basis from which the quaternion distribution theory is further developed, and will be discussed in Section 3.1.

In Section 3.2 we derive the probability density and characteristic function of the matrix-variate quaternion normal distribution using the real representation thereof.

3.1 The p-variate quaternion normal distribution

In this section the approach of Kabe ([7], [8], [9]), Rautenbach [13] and Rautenbach and Roux ([14], [15]) are followed in deriving the p-variate quaternion normal distribution. Although the results in this section 3.1 are in general not new, it is shown how they relate to those given by Teng and Fang [16].

Definition 2 Let

$$\underset{p\times 1}{\underline{Z}} = [Z_1,\ldots,Z_p]' = \begin{bmatrix} X_{11}+iX_{21}+jX_{31}+kX_{41} \\ X_{12}+iX_{22}+jX_{32}+kX_{42} \\ \vdots \\ X_{1p}+iX_{2p}+jX_{3p}+kX_{4p} \end{bmatrix}$$

be a quaternion probability vector with real associated probability vector

$$\underset{4p\times 1}{\underline{Z}_0} = [X_{11},\ldots,X_{1p},X_{21},\ldots,X_{2p},X_{31},\ldots,X_{3p},X_{41},\ldots,X_{4p}]' = \left[\underset{1\times p}{\underline{X}'_1}, \underset{1\times p}{\underline{X}'_2}, \underset{1\times p}{\underline{X}'_3}, \underset{1\times p}{\underline{X}'_4}\right]'.$$

Then, $\underline{Z}$ has a p-variate quaternion normal distribution if $\underline{Z}_0$ has a $4p$-variate real normal distribution.

Teng and Fang [16] used a different matrix structure for representing quaternions by matrices. They supposed that $\underset{p\times 1}{\underline{Z}} = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4$ may be represented by

$$\underset{4p\times 4}{\mathbf{Z}_{00}} = \begin{bmatrix} \underset{p\times 1}{\underline{X}_1} & \underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_3} & \underset{p\times 1}{\underline{X}_4} \\ -\underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} & -\underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_3} \\ -\underset{p\times 1}{\underline{X}_3} & \underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_1} & -\underset{p\times 1}{\underline{X}_2} \\ -\underset{p\times 1}{\underline{X}_4} & -\underset{p\times 1}{\underline{X}_3} & \underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} \end{bmatrix} = \left[\underset{4p\times 1}{\underline{Y}_1}, \underset{4p\times 1}{\underline{Y}_2}, \underset{4p\times 1}{\underline{Y}_3}, \underset{4p\times 1}{\underline{Y}_4}\right].$$

Thus the conjugate $\underset{p\times 1}{\bar{\underline{Z}}} = \underline{X}_1 - i\underline{X}_2 - j\underline{X}_3 - k\underline{X}_4$ of $\underset{p\times 1}{\underline{Z}}$ may be represented as

$$\underset{4p\times 4}{\bar{\mathbf{Z}}_{00}} = \begin{bmatrix} \underset{p\times 1}{\underline{X}_1} & -\underset{p\times 1}{\underline{X}_2} & -\underset{p\times 1}{\underline{X}_3} & -\underset{p\times 1}{\underline{X}_4} \\ \underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} & \underset{p\times 1}{\underline{X}_4} & -\underset{p\times 1}{\underline{X}_3} \\ \underset{p\times 1}{\underline{X}_3} & -\underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_1} & \underset{p\times 1}{\underline{X}_2} \\ \underset{p\times 1}{\underline{X}_4} & \underset{p\times 1}{\underline{X}_3} & -\underset{p\times 1}{\underline{X}_2} & \underset{p\times 1}{\underline{X}_1} \end{bmatrix} = \left[\underset{4p\times 1}{\underline{Y}^*_1}, \underset{4p\times 1}{\underline{Y}^*_2}, \underset{4p\times 1}{\underline{Y}^*_3}, \underset{4p\times 1}{\underline{Y}^*_4}\right].$$

From this it is clear that $\underline{Z}_0 = \underline{Y}^*_1$. Teng and Fang [16] showed that any of the $\underline{Y}_s$, $s = 1,2,3,4$ may be used to arive at the same form of the probability density function for the p-variate quaternion normal distribution.

The covariance matrix $\mathbf{\Sigma}$ of $\underline{Z}$ relates to its real associated covariance matrix $\mathbf{\Sigma}_0$, i.e. the covariance matrix of $\underline{Z}_0$, in the following way:

$$(6)\quad \underset{4p\times 4p}{\mathbf{\Sigma}_0} = \operatorname{cov}\left(\underline{Z}_0, \underline{Z}_0'\right) = E\left[\left(\underline{Z}_0 - \underline{\mu}_{\underline{Z}_0}\right)\left(\underline{Z}_0 - \underline{\mu}_{\underline{Z}_0}\right)'\right] = \frac{1}{4}\begin{bmatrix} \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_2 & -\mathbf{\Sigma}_3 & -\mathbf{\Sigma}_4 \\ \mathbf{\Sigma}_2 & \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_4 & \mathbf{\Sigma}_3 \\ \mathbf{\Sigma}_3 & \mathbf{\Sigma}_4 & \mathbf{\Sigma}_1 & -\mathbf{\Sigma}_2 \\ \mathbf{\Sigma}_4 & -\mathbf{\Sigma}_3 & \mathbf{\Sigma}_2 & \mathbf{\Sigma}_1 \end{bmatrix}$$
$$\simeq \frac{1}{4}\left(\mathbf{\Sigma}_1 + i\mathbf{\Sigma}_2 + j\mathbf{\Sigma}_3 + k\mathbf{\Sigma}_4\right) = \frac{1}{4}\operatorname{cov}\left(\underline{Z}, \underline{Z}'\right)$$

where $\underset{p\times p}{\mathbf{\Sigma}_1}$ is a real symmetric matrix, and $\underset{p\times p}{\mathbf{\Sigma}_2}$, $\underset{p\times p}{\mathbf{\Sigma}_3}$ and $\underset{p\times p}{\mathbf{\Sigma}_4}$ are skew symmetric matrices. Returning to the $\underline{Y}_s$, $s = 1, 2, 3, 4$ defined by Teng and Fang [16], they showed that $\frac{1}{4}\mathrm{cov}(\underline{Z}, \underline{Z}') \simeq \mathrm{cov}(\underline{Y}_s, \underline{Y}_s')$, $s = 1, 2, 3, 4$.

In order to apply the representation theory discussed in Section 2, i.e. by an elementwise expansion of the quaternion probability vector, we now rearrange the components of the real associated probability vector as follows:

$$\underset{4p\times 1}{\underline{Z}_0^*} = \left[X_{11}, X_{21}, X_{31}, X_{41}, \ldots, X_{1p}, X_{2p}, X_{3p}, X_{4p}\right]'.$$

The components of $\underline{\mu}_0$ and $\mathbf{\Sigma}_0$ are rearranged accordingly in forming $\underset{4p\times 1}{\underline{\mu}_0^*}$ and $\underset{4p\times 4p}{\mathbf{\Sigma}_0^*}$ respectively, and now yield the pdf of $\underline{Z}_0^*$ as

$$(7)\quad f_{\underline{Z}_0^*}(\underline{z}_0^*) = (2\pi)^{-\frac{1}{2}(4p)}\left\{\det \mathbf{\Sigma}_0^*\right\}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\underline{z}_0^* - \underline{\mu}_0^*\right)' \mathbf{\Sigma}_0^{*-1}\left(\underline{z}_0^* - \underline{\mu}_0^*\right)\right\}$$

for $\underline{z}_0^* \in B_0^* = \left\{\underline{z}_0^* = \left[x_{11}, x_{21}, x_{31}, x_{41}, \ldots, x_{1p}, x_{2p}, x_{3p}, x_{4p}\right]', -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty, s = 1, \ldots, p\right\}$. Since $f_{\underline{Z}_0}(\underline{z}_0) = f_{\underline{Z}_0^*}(\underline{z}_0^*)$ for all $\underline{z}_0$ and corresponding $\underline{z}_0^*$, we may use $\underline{Z}_0^*$ as real associated probability vector when deriving the pdf of $\underline{Z}$.

Theorem 3 (See Rautenbach and Roux [15].) Let $\underset{p\times 1}{\underline{Z}}$ be a probability vector that has a p-variate quaternion normal distribution, with $E[\underline{Z}] \equiv \underline{\mu}$ and $\mathbf{\Sigma} \equiv [\sigma_{st}]$ as given in (6). The pdf of $\underset{p\times 1}{\underline{Z}} \sim \mathbb{Q}N(p; \underline{\mu}, \mathbf{\Sigma})$, is given by:

$$f_{\underline{Z}}(\underline{z}) = 2^{2p}\pi^{-2p}\left(\det \mathbf{\Sigma}\right)^{-2} \exp\left\{-2\left(\overline{\underline{z} - \underline{\mu}}\right)' \mathbf{\Sigma}^{-1}\left(\underline{z} - \underline{\mu}\right)\right\}$$

where $\underline{z} \in B = \left\{\underline{z} = [z_1, \ldots, z_p]' : z_s = x_{1s} + ix_{2s} + jx_{3s} + kx_{4s}, -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty, s = 1, \ldots, p\right.$

Theorem 4 (See Rautenbach and Roux [15].) The characteristic function of $\underset{p\times 1}{\underline{Z}} \sim \mathbb{Q}N(p; \underline{\mu}, \mathbf{\Sigma})$, is given by:

$$(8)\quad \phi_{\underline{Z}}(\underline{t}) = \exp\left\{\frac{\iota}{2}\left(\underline{\bar{\mu}}'\underline{t} + \underline{\bar{t}}'\underline{\mu}\right) - \frac{1}{8}\underline{\bar{t}}'\mathbf{\Sigma}\underline{t}\right\}$$

for every quaternion vector $\underset{p\times 1}{\underline{t}}$ and ι the usual imaginary complex unit.

3.2 The matrix-variate quaternion normal distribution

We now turn our attention to the derivation of the matrix-variate quaternion normal distribution by expanding our results of the previous section. Once again our goal is to emphasise the relationship between the *real associated form* and its resultant counterpart. The relationship between the characteristic function of the matrix-variate quaternion normal distribution and its real associated counterpart will be established.

Theorem 5 Let $\underset{p\times 1}{\underline{Z}_\alpha}, \alpha = 1, \ldots, n$ be n probability vectors each having a p-variate quaternion normal distribution. Now, suppose (for $\alpha = 1, \ldots, n,\ \beta = 1, \ldots, p$) that

$$\underset{n\times p}{\mathbf{Z}} = [Z_{\alpha\beta}] = \begin{bmatrix} Z_{11} & \ldots & Z_{1p} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \ldots & Z_{np} \end{bmatrix} = \begin{bmatrix} \underset{1\times p}{\underline{Z}'_1} \\ \vdots \\ \underset{1\times p}{\underline{Z}'_n} \end{bmatrix} = \left[\underset{n\times 1}{\underline{Z}_{(1)}}, \ldots, \underset{n\times 1}{\underline{Z}_{(p)}} \right]$$

i.e. the rows of $\mathbf{Z}$ are $\mathbb{Q}N(p; \underline{\mu}_\alpha, \mathbf{\Sigma})$ distributed, $\alpha = 1, \ldots, n$ with dependence structure given by $\underset{n\times n}{\mathbf{R}}$ not necessarily equal to $\mathbf{I}_n$. It may be assumed without loss of generality that $\mathbf{R}$ is real-valued. Similarly, define

$$\underset{n\times p}{\mu} = [\mu_{\alpha\beta}]\,, \ \alpha = 1, \ldots, n,\ \beta = 1, \ldots, p.$$

Then

$$\underset{np\times 1}{\text{vec}\mathbf{Z}} = \begin{bmatrix} \underset{n\times 1}{\underline{Z}_{(1)}} \\ \vdots \\ \underset{n\times 1}{\underline{Z}_{(p)}} \end{bmatrix} \sim \mathbb{Q}N\left(np; \text{vec}\mu, \mathbf{\Sigma} \otimes \mathbf{R}\right)$$

i.e. a matrix-variate quaternion normal distribution where $\underset{np\times 1}{vec\mu} = \begin{bmatrix} \underset{n\times 1}{\underline{\mu}_{(1)}} \\ \vdots \\ \underset{n\times 1}{\underline{\mu}_{(p)}} \end{bmatrix}$

(denote $\underset{n\times p}{\mathbf{Z}} \sim \mathbb{Q}N\left(n \times p; \mu, \mathbf{\Sigma} \otimes \mathbf{R}\right)$).

Proof. This proof will form part of the presentation or may be obtained from the authors.

■

We now establish the relationship between the characteristic function of the matrix-variate quaternion normal distribution and that of its real associated matrix-variate normal distribution.

Theorem 6 Let $\underset{n\times p}{\mathbf{Z}} \sim \mathbb{Q}N\left(n \times p; \mu, \mathbf{\Sigma} \otimes \mathbf{R}\right)$ as defined above. The characteristic function of $\mathbf{Z}$ is given by

$$(9) \quad \phi_{\mathbf{Z}}(\mathbf{T}) = \exp \text{Re}\, tr \left\{ \iota\bar{\mu}'\mathbf{T} - \frac{1}{8}\mathbf{\Sigma}\bar{\mathbf{T}}'\mathbf{R}\mathbf{T} \right\}$$

where $\underset{n\times p}{\mathbf{T}} \in M_{n\times p}(\mathbb{Q})$ and where ι is the usual complex unit.

Proof. This proof will form part of the presentation or may be obtained from the authors.

■

Note that $\phi_{\mathbf{Z}}(\mathbf{T}) = \exp \text{Re}tr\left\{\iota\bar{\mu}'\mathbf{T} - \frac{1}{8}\mathbf{\Sigma}\bar{\mathbf{T}}'\mathbf{R}\mathbf{T}\right\} = \exp tr \left\{\iota\mu_0^{*\prime}\mathbf{T}_0^* - \frac{1}{2}\mathbf{\Sigma}_0^*\mathbf{T}_0^{*\prime}\mathbf{R}\mathbf{T}_0^*\right\} = \phi_{\mathbf{Z}_0^*}(\mathbf{T}_0^*)$, satisfying (5).

4 Applications illustrating the role of the quaternion normal distribution

Is it possible to find the density function of the quaternion Wishart matrix from the real associated Wishart matrix?

In this section, questions that generally arise in distribution theory will be addressed. The quaternion Wishart distribution will be derived from the real matrix-variate normal distribution associated with the matrix-variate quaternion normal distribution by which it is defined. We once again emphasise the link between the characteristic functions of the quaternion and real associated Wishart distributions.

Kabe ([7], [8], [9]) derived the hypercomplex Wishart distribution directly from the hypercomplex normal distribution using the Q generalized Sverdrup's lemma. Teng and Fang [16] showed that the maximum likelihood estimator $\hat{\mathbf{\Sigma}}$ of $\mathbf{\Sigma}$ followed a quaternion Wishart distribution. They used a Fourier transform on the results given by Andersson[1] to yield explicit expressions for the probability density and characteristic functions of the quaternion Wishart distribution. The non-central quaternion Wishart distribution was discussed by Kabe [9], while Li and Xue [11] derived the singular quaternion Wishart distribution. More technical results, specifically regarding Selberg-type squared matrices, gamma and beta integrals are found in the paper by Gupta and Kabe [6].

Theorem 7 Let $\underset{n\times p}{\mathbf{Z}} \sim \mathbb{Q}N\left(n\times p; \mathbf{0}, \mathbf{\Sigma}\otimes \mathbf{I}_n\right)$. Then for $n \geq p$, $\mathbf{W} = \bar{\mathbf{Z}}'\mathbf{Z}$ is said to have the quaternion Wishart distribution with n degrees of freedom, i.e. $\mathbf{W} \sim \mathbb{Q}W_p\left(n, \mathbf{\Sigma}\right)$, with density function given by

$$(10)\quad f(\mathbf{W}) = \frac{2^{2np}}{\mathbb{Q}\Gamma_p(2n)\left(\det \mathbf{\Sigma}\right)^{2n}} \exp\left\{-2\,\mathrm{Re}\ \mathrm{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W}\right)\right\} \det\left(\mathbf{W}\right)^{2n-2p+1},$$

with $\mathbf{W} = \bar{\mathbf{W}}' > \mathbf{0}$ and where $\mathbb{Q}\Gamma_p\left(\cdot\right)$ is the multivariate quaternion gamma function, as given in [5].

Proof This proof will form part of the presentation or may be obtained from the authors.

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