Copulae for High-Frequency Data

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Introduction

The modelling of the realized volatility and covariances has became a corner-stone in financial risk management and portfolio optimization. The first brick in intraday modelling has been laid by Andersen and Bollerslev (1998). Afterwards a series of papers providing estimators of the realized volatility appeared and a detailed investigation has been provided. More recently the attention shifted towards modelling the joint behavior of returns. A variety of time-series models with Gaussian innovations have thus been proposed. Apart of modelling volatility of returns separately, Andersen, Bollerslev, Diebold and Labys (1999) introduce the notion of realized correlation and realized covariance which helps to model high dimensional high frequency joint behaviour. Recent works dedicated to modelling the realized covariance/correlation matrix are by Hayashi and Yoshida (2005) and Audrino and Corsi (2009). Most crucial point in these papers, the residuals are assumed to be normal and the dependency is that of a normal structure. The Gaussian structure leads to imprecise modelling of the joint tails of the distribution as it has no upper nor lower tail dependency. Recent evidence shows, that a series of financial data are not following normal dependency, Lee and Long (2009). For static, non time dependent models a normal distribution can be replaced by the copula function, see Sklar (1959), Nelsen (2006), etc. Modelling with time varying copulae based innovations is increasing in popularity, see Patton (2004), Chen ad Fan (2006), Härdle, Giacomini, Spokoiny (2009), Härdle, Okhrin, Okhrin (2011), etc. To the authors’ knowledge the only paper that is using intraday data to model the dependency is Breymann, Dias and Embrechts (2003). In this paper we propose a novel approach of time varying copulae using intraday data. Current work discusses different models of realized covariances and appropriate realized variances. The quality of this approach is compared by the out of sample forecast using an adaptive estimation of the copula model, and the simple rolling window applied to the residuals from some simple time varying model.

The paper is organized as follows. First we provide a discussion on different methods of estimation of the realized volatility and covariances. The second section deals with copula theory, mainly with the recent time varying models. The Section 3 states the main result of our model. A simulation study and an empirical part follows.

Realized volatility and correlation
In the paper we consider a $d$-dimensional process of log-prices $P = (P_1, \ldots, P_d)^\top$ observed on the interval $[0; T]$, where $T$ is considered to be an integer number measured in days. The observation times for the asset $j$ at day $t$ are denoted by $t_{j,t,1}, t_{j,t,2}, \ldots$ and are assumed to be strictly increasing, from which follows that our observations are given by $P_{j,t,i} = P_{j,t_{j,t,i}}$, $j = 1, \ldots, d$, $i = 1, \ldots, N_j$, $t = 1, \ldots, T$. The observation points differ for all processes which leads to a different number of observations. The observed log-price process $P$ is assumed to be driven by the efficient price process $Y$, which in our setup is a Brownian semimartingale with a jump component (BSMJ)

$$dY_t = \mu_t dt + \sigma_t dW_t + dJ_t$$

with $\mu_t$ being vector of predictable drifts, and $\sigma_t$ cadlag volatility matrix process, and $W_t$ is a vector of independent Brownian motions. The market microstructure effect is modeled through an additive function $C$ of random variables $X$.

The main result states that if $F$ is the impact of the marginal distributions. Formally copulae where introduced in Sklar (1959). The margins and a pure dependency component it captures the dependency between variables eliminating the impact of the marginal distributions. For the estimation of $[Y]$ from discrete, non-synchronous and noisy price observations, we will use the realized kernel method introduced by Barndorff-Nielsen, Hansen, Lunde, Shephard (BNHLS) (2008) and is considered to be consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading. The multivariate realized kernel is defined as

$$K(P) = \sum_{h=-H}^{H} k\left(\frac{|h|}{H+1}\right) \Gamma_h,$$

with $\Gamma_h$ being a matrix of autocovariances given by

$$\Gamma_h = \begin{cases} 
\sum_{j=|h|+1}^{n} p_j p_{j-h}^\top, & h \geq 0 \\
\sum_{j=|h|+1}^{n} p_j p_{j-h}^\top, & h < 0 
\end{cases},$$

and $k(x)$ being the weight function of the Parzen kernel. For a discussion on the bandwidth selection we refer to the web appendix of BNHLS (2008).

**Time Varying Copula**

The advantage of the copula is that it allows to split the multivariate distribution into the margins and a pure dependency component it captures the dependency between variables eliminating the impact of the marginal distributions. Formally copulae where introduced in Sklar (1959). The main result states that if $F$ is an arbitrary $d$-dimensional continuous distribution function of the random variables $X_1, \ldots, X_d$, then the associated copula is unique and defined as the continuous function $C : [0,1]^d \rightarrow [0,1]$ which satisfies the equality

$$C(u_1, \ldots, u_d) = F[F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)], \quad u_1, \ldots, u_d \in [0,1],$$

where $F_1^{-1}(\cdot), \ldots, F_d^{-1}(\cdot)$ are the quantile functions of the corresponding marginal distributions. One of the classes that overcomes the drawback of elliptical copulae of having no explicit form of the cdf is the class of Archimedean copulae

$$C(u_1, \ldots, u_k) = \phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d), \quad u_1, \ldots, u_d \in [0,1],$$

where $\phi \in \mathcal{L} = \{ \phi : [0; \infty) \rightarrow [0,1] | \phi(0) = 1, \phi(\infty) = 0; (-1)^j \phi^{(j)} \geq 0; j = 1, \ldots, \infty \}$. The function $\phi$ is called the generator of the copula and commonly depends on a single parameter $\theta$. A detailed review of the properties of Archimedean copulae can be found in McNeil and Neslehova (2008).
In this paper we consider a classical rolling window approach (with the window width \( w \)) and the approach based on a local constant copula approximation. Corresponding theory and applications may be found in Giacomini, Härdle, Spokoiny (2009) and Cizek, Härdle, Spokoiny (2009).

Let \( \theta_t \) denote the time varying but unknown copula parameter. The idea is to select for each time point \( t_0 \) an interval \( I \) at which \( \theta_t \) can be well approximated by a constant \( \theta^* \). The discrepancy between two copulae \( C(\cdot; \theta) \) and \( C(\cdot; \theta^*) \) is measured by the Kullback-Leibler divergence \( K(C(\cdot; \theta), C(\cdot; \theta^*)) = E_{\theta^*} \log \frac{C(\cdot; \theta)}{C(\cdot; \theta^*)} \), where \( c \) is the copula density. The aim is to select \( I \) as close as possible to the so-called “oracle” choice interval \( I_k^* \), defined as the largest interval \( I = [t_0 - m_k; t_0] \), for which the small modelling bias condition (SMB) is fulfilled, i.e. \( \Delta I(\theta) = \sum_{t \in I} K(C(\cdot; \theta), C(\cdot; \theta)) \leq \Delta \), for some \( \Delta \geq 0 \), \( \theta \). The LCP is based on sequentially testing the hypotheses of homogeneity on intervals \( I_k \).

We select \( I_k \) with \( k = -1, 0, 1, \ldots \) as the sequence of intervals \( I_k \subset I_{k+1} \), starting with \( k = 1 \). If there are no change points in \( T_k \subset I_k \setminus I_{k-1} \) we accept \( I_k \) as an interval with constant copula parameter and structure. At the next step we take \( T_{k+1} \) and test it for homogeneity. We repeat the steps until rejection or the largest possible interval \( I_K \) is accepted, leading to an interval \( I_k^* \).

**Realized Copula**

From the Hoeffdings lemma it is known that the covariance \( \sigma_{ij} \) for random variables \( X_i \) and \( X_j \) with marginal distributions \( F_i \) and \( F_j \), with finite first and second moments and the copula \( C_\theta \) is given by:

\[
\sigma_{ij}(\theta) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} C_\theta(F_i(x_i), F_j(x_j)) - F_i(x_i)F_j(x_j) dx_i dx_j
\]

Following theory of the intraday data we suppose the daily returns \( r_t \sim N(0; \sqrt{IV}) \)

Usually this integral has no explicit form, except as for the normal distribution, where one gets \( \sigma = \theta \). For \( d = 2 \) one has to invert \( \sigma_{ij}(\theta) \) to get the estimator of the copula parameter \( \theta_{ij} = \sigma_{ij}^{-1}(\theta) \). This can be problematic, when one deals with a high dimensional problem, simple Archimedean copulae depend only on the one parameter, thus, there should be a method, allowing some approximation of the copula parameter. Let us first define the function which measures losses from the estimator of the volatility, without any model assumption (in the copula sense) and based on the copula: \( g_{ij}(\theta) = \sigma_{ij} - \sigma_{ij}(\theta) \). Here we propose several estimators of the copula parameter based on the realized covariance

1. An average of the all possible copula parameters based on the pairs

\[
\hat{\theta}_{av} = \frac{2}{d(d-1)} \sum_{i \leq j} \hat{\theta}_{i,j}
\]

2. For the second method we propose one-stage method of moments estimator with the identity weight matrix \( W = I_{d \times d} \). By denoting \( g(\theta) = \{g_{ij}(\theta)\}_{i \leq j}^\top \) the estimator is given by

\[
\theta_{GMM} = \arg\min_{\theta} g^\top(\theta)g(\theta)
\]

This method is equivalent to the first one in the two dimensional case.

3. The two-stage GMM estimator with \( W = \{E[g(\theta_{GMM})g^\top(\theta_{GMM})]\}^{-1} \), given by

\[
\hat{\theta}_{GMM} = \arg\min_{\theta} g^\top(\theta)Wg(\theta)
\]
Table 1: Simulation results for the 3-dimensional realized copula using the factor SV model with nonsynchronous observations and measurements noise. Bias and root mean square error are reported.

<table>
<thead>
<tr>
<th></th>
<th>HL mean</th>
<th>R.mse</th>
<th>HLgmm mean</th>
<th>R.mse</th>
<th>LinCor mean</th>
<th>R.mse</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi = 0, \lambda = (30, 45, 60)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$RC_{390}$</td>
<td>1.349</td>
<td>0.651</td>
<td>1.340</td>
<td>0.662</td>
<td>1.356</td>
<td>0.645</td>
</tr>
<tr>
<td>$RC_{26}$</td>
<td>1.929</td>
<td>0.300</td>
<td>1.980</td>
<td>0.364</td>
<td>1.938</td>
<td>0.286</td>
</tr>
<tr>
<td>$K$</td>
<td>1.843</td>
<td>0.199</td>
<td>1.856</td>
<td>0.214</td>
<td>1.836</td>
<td>0.201</td>
</tr>
<tr>
<td>$\xi = 0.001, \lambda = (30, 45, 60)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$RC_{390}$</td>
<td>1.176</td>
<td>0.824</td>
<td>1.170</td>
<td>0.830</td>
<td>1.183</td>
<td>0.817</td>
</tr>
<tr>
<td>$RC_{26}$</td>
<td>1.842</td>
<td>0.320</td>
<td>1.907</td>
<td>0.382</td>
<td>1.857</td>
<td>0.306</td>
</tr>
<tr>
<td>$K$</td>
<td>1.667</td>
<td>0.348</td>
<td>1.706</td>
<td>0.321</td>
<td>1.668</td>
<td>0.346</td>
</tr>
</tbody>
</table>

Empirical findings show, that this is almost equivalent to the first method.

Simulation Study

In this simulation study we considered almost the same model as in BNHLS(2011) with the only one difference of the copula based dependence:

\[
X = Y + U, \quad dY = \mu dt + \sigma dW_c
\]

\[
\sigma^{(i)} = \exp\{\beta_0 + \beta_1 \varrho^{(i)}(t)\} \quad \text{with} \quad d\varrho^{(i)} = \alpha \varrho^{(i)} dt + dB^{(i)} \quad \text{and} \quad \varrho^{(i)}(0) = N(0, (\beta_0 - 2\alpha)^{-1})
\]

\[
U_j|Y \sim N(0, \omega) \quad \text{with} \quad \omega^2 = \xi^2 \sqrt{N^{-1} \sum_{j=1}^{N} \sigma^{(i)}(j/N)}
\]

\[
(\mu, \beta_0, \beta_1, \alpha, \varrho, \theta, \xi^2) \quad = \quad (0.03, -5/16, 1/8, -1/40, -0.3, 2.0, \{0, 0.001\})
\]

where $dW_c$ is the copula dependent vector of Brownian motions, what means, that increments are copula dependent with parameter $\theta$. $U$ is an additive noise, and $\xi^2$ is the signal-to-noise ratio. For the simulation we apply Euler scheme with the sample size $N = 23400$ which corresponds to trading every second in 6.5 hours. To introduce nonsynchronous trading we implement $d$ independent Poisson processes with intensity $\lambda$, for $d = 2$ we use $\lambda = (30, 45, 60)$. Bias and root mean square error for all models in two and three dimensional case are provided in the Table 1.

Empirical Study

The empirical part of this work is based on data taken from NYSE’s Trades and Quotes (TAQ) database, for Citigroup (C), IBM and Alcoa (AA) for the period from 01.01.2007 till 31.07.2009 which
Figure 1: Estimation of the copula parameter based on all methods discussed above.

covers the very turbulent periods of the financial crisis in the mid of 2008. This period covers 638 days in total and daily transactions from 9:30 till 16:00. We apply the cleaning procedure proposed by BNHLS (2008).

Let us denote by $w$ the portfolio, which is represented by the number of assets for a specified stock, $w = \{w_1, \ldots, w_d\}$, $w_i \in \mathbb{Z}$. The value $V_t$ of the portfolio $w$ is given non-recursive by $V_t = \sum_{j=1}^{d} w_j S_{j,t}$ and the profit and loss function by $L_{t+1} = \sum_{j=1}^{d} w_j S_{j,t} \{\exp(X_{j,t+1}) - 1\}$. For every model we simulate returns assuming no autocorrelation, by taking the respective dependency parameter (realized copula, copula from LCP or rolling window) and appropriate volatility. If we take the rolling window or LCP method, volatility is just the variance of the returns taken from the respective interval. For the realized copula volatility is a realized volatility. The distribution function of $L$, dropping the time index, is given by $F_L(x) = P(L \leq x)$. As usual the VaR at level $\alpha$ from a portfolio $w$ is defined as the $\alpha$-quantile from $F_L$ i.e. $\text{VaR}(\alpha) = F_L^{-1}(\alpha)$. $L$ depends on the $d$-dimensional distribution of log-returns $F_X$. Thus, modelling their distribution is essential to obtain the quantiles from $F_L$.

 Afterwards backtesting is used to evaluate the performance of the specified copula family $C$. The estimated values for the VaR are compared with the true realisations $\{l_t\}$ of the P&L function, an exceedance occurring for each $l_t$ smaller than $\hat{\text{VaR}}_t(\alpha)$. The ratio of the number of exceedances to the number of observations gives the exceedance ratio $\hat{\alpha}$: $\hat{\alpha} = \frac{1}{T-r} \sum_{t=r}^{T} I \{ l_t < \hat{\text{VaR}}_t(\alpha) \}$.

REFERENCES (RÉFÉRENCES)


Table 2: \( \hat{\alpha} \) for different models, for given \( \alpha \in \{0.01, 0.05, 0.1\} \). With bold we represent closest \( \hat{\alpha} \) for given \( \alpha \). The study has been done for \( 10^6 \) simulations. In the brackets we represent the Kupiec test statistic, which should \( \chi^2 \) distributed.

<table>
<thead>
<tr>
<th>Model</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>lcp ((m_0 = 20))</td>
<td>0.0470 (46.536)</td>
<td>0.095 (22.308)</td>
<td>0.134 (7.827)</td>
</tr>
<tr>
<td>lcp ((m_0 = 40))</td>
<td>0.0564 (66.756)</td>
<td>0.100 (26.654)</td>
<td>0.145 (13.204)</td>
</tr>
<tr>
<td>rol ((w = 25))</td>
<td>0.0297 (16.482)</td>
<td>0.087 (15.798)</td>
<td>0.123 (7.827)</td>
</tr>
<tr>
<td>rol ((w = 50))</td>
<td>0.0376 (28.850)</td>
<td>0.086 (14.611)</td>
<td>0.125 (4.265)</td>
</tr>
<tr>
<td>rol ((w = 100))</td>
<td>0.0454 (43.399)</td>
<td>0.089 (17.024)</td>
<td>0.134 (7.827)</td>
</tr>
<tr>
<td>HLgmm</td>
<td>0.0266 (12.261)</td>
<td>0.086 (14.611)</td>
<td>0.139 (9.975)</td>
</tr>
<tr>
<td>HL</td>
<td>0.0297 (16.482)</td>
<td>0.089 (17.024)</td>
<td>0.144 (12.358)</td>
</tr>
<tr>
<td>K</td>
<td>0.0282 (14.314)</td>
<td>0.087 (15.798)</td>
<td>0.142 (11.538)</td>
</tr>
<tr>
<td>Gauss</td>
<td><strong>0.0031</strong> (4.150)</td>
<td><strong>0.023</strong> (11.630)</td>
<td>0.059 (13.361)</td>
</tr>
</tbody>
</table>


