Noncentral limit theorems for statistical functionals based on long-memory sequences

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Introduction

Let $V_T$ be a class of distribution functions on the real line, $V'$ be a vector space (e.g., $V' = \mathbb{R}$), and $T : V_T \to V'$ be a statistical functional. Let $(X_t)_{t \in \mathbb{N}}$ be a strictly stationary sequence of random variables with distribution function $F \in V_T$. If $\hat{F}_n$ denotes the empirical distribution functions of $X_1, \ldots, X_n$, i.e., $\hat{F}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{[X_i, \infty)}$, then $T(\hat{F}_n)$ can provide a reasonable estimator for $T(F)$. In the context of nonparametric statistics, a central question concerns the asymptotic distribution of $T(\hat{F}_n)$. On the one hand, in the case of weakly dependent observations $X_1, X_2, \ldots$ satisfying certain mixing conditions, there are several general results on the asymptotic distribution of $T(\hat{F}_n)$ for various functionals $T$. See, for instance, [4, 18] for L-functionals, and [5, 10, 11, 25] for V-functionals. On the other hand, in the case of strongly dependent observations $X_1, X_2, \ldots$, whose appearance has been observed in numerous scientific areas [2, 3, 20], there are only a few results on the asymptotic distribution of the plug-in estimator $T(\hat{F}_n)$ for some selected functionals $T$; see, for instance, [9, 17].

This paper (based on [4, 5, 6]) is concerned with a unifying approach for deriving the asymptotic distribution of $T(\hat{F}_n)$ for strongly dependent data. We will avail a version of the Functional Delta Method (FDM). The latter allows to derive the asymptotic distribution of the plug-in estimator $T(\hat{F}_n)$ from the asymptotic distribution of the empirical distribution function $\hat{F}_n$ as long as the functional $T$ is sufficiently regular, more precisely, Hadamard differentiable. The classical FDM [13, 14, 19] was repeatedly criticized for its restricted range of applications. Many tail-dependent statistical functionals $T$ (e.g. general L- or V-functionals) are known to be non-Hadamard differentiable at $F$. However, recently the concept of quasi-Hadamard differentiability was introduced in [4]. This is a weaker concept of differentiability (in particular general L- and V-functionals can be shown to be quasi-Hadamard differentiable), but it is still strong enough to obtain an FDM (referred to as Modified FDM); cf. [4, Section 4]. The basic idea of quasi-Hadamard differentiability is to impose a norm only on a suitable subspace $V_0$ of the space $D` (\supseteq V_T)$ of all bounded càdlàg functions on $\mathbb{R}$ (and not on all of $D`$), and to differentiate only in directions which lie in (some subset of) $V_0$. It should be stressed that this is not simply the notion of tangential Hadamard-differentiability [13, 14, 19] where the tangential space is equipped with the same norm as $D`$. The crucial point is that norms, which assign to $F$ a finite length, are often not strict enough to obtain “differentiability”. On the other hand, “differentiability” w.r.t. such good-natured norms is typically not necessary. For details the reader is referred to [4, Section 1].

Upon having established quasi-Hadamard differentiability of a given statistical functional $T$, an application of the Modified FDM typically requires weak convergence of the underlying empirical process w.r.t. a norm being stricter than the sup-norm $\| \cdot \|_\infty$, for instance w.r.t. a weighted sup-norm $\| \cdot \|_\lambda := \| (\cdot) \phi_\lambda \|_\infty$ with $\phi_\lambda(x) := (1+|x|)^\lambda$ for some $\lambda > 0$. Here $\lambda$ depends on the considered statistical functional $T$. Hence in the context of strongly dependent data, the crucial point is a Noncentral Limit Theorem (NCLT) for the empirical distribution function in a weighted sup-norm. We will first of all present such a result; cf. Theorem 1. Corresponding CLTs can be found in [22] for independent data, in [7] for weakly dependent $\beta$-mixing data, in [21] for weakly dependent $\alpha$- and $\rho$-mixing data, and in [24] for weakly dependent causal data.
NCLT for the empirical distribution function of long-memory sequences

Let

\[ X_t := \sum_{s=0}^{\infty} a_s \varepsilon_{t-s}, \quad t \in \mathbb{N}, \]

where \((\varepsilon_t)_{t \in \mathbb{Z}}\) are i.i.d. random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with zero mean and finite variance, and the coefficients \(a_s\) satisfy \(\sum_{s=0}^{\infty} a_s^2 < \infty\) (so that \((X_t)_{t \in \mathbb{Z}}\) is an \(L^2\)-process). We assume that the sequence \((X_t)_{t \in \mathbb{Z}}\) is strictly stationary with distribution function \(F\). Many important time series models, such as ARMA and FARIMA, take this form. If \(a_0 = 1\) and \(a_1 = a_2 = \cdots = 0\), then the \(X_t\) are i.i.d. If \(a_t\) decays to zero at a sufficiently fast rate, then the covariances \(\text{Cov}(X_1, X_t)\) are summable over \(t \in \mathbb{Z}\) and thus the process exhibits short-range dependence (weak dependence). If \(a_t\) decays to zero at a sufficiently slow rate, then the covariances \(\text{Cov}(X_1, X_t)\) are not summable over \(t \in \mathbb{N}\) and thus the process exhibits long-range dependence (strong dependence).

If the \(X_t\) are i.i.d., then it is commonly known that the empirical process \(n^{1/2}(\hat{F}_n - F)\) converges in distribution to an \(F\)-Brownian bridge, i.e. to a centered Gaussian process with covariance function \(\Gamma(s,t) = F(s \wedge t)F(s \vee t)\). If the \(X_t\) are subject to a certain mixing condition (weak dependence), then the limit in distribution of the empirical process \(n^{1/2}(\hat{F}_n - F)\) is known to be a centered Gaussian process with covariance function \(\Gamma(s,t) = F(s \wedge t)F(s \vee t) + \sum_{k=2}^{\infty} \text{Cov}(\mathbb{1}_{\{X_1 \leq s\}}, \mathbb{1}_{\{X_k \leq t\}}) + \text{Cov}(\mathbb{1}_{\{X_1 \leq \ell\}}, \mathbb{1}_{\{X_2 \leq s\}})\); see [7, 12, 21, 24]. If the \(X_t\) exhibit long-range dependence (strong dependence, long-memory), then the situation changes drastically: Assuming a moving average structure (1) with \(a_s = s^{-\beta}, \ s \in \mathbb{N}\), for \(\beta \in (\frac{1}{2}, 1)\), and some additional regularity and moment conditions on the distribution of \(\varepsilon_0\), one has

\[ n^{\beta-1/2}(\hat{F}_n(\cdot) - F(\cdot)) \xrightarrow{d} c_\beta f(\cdot)Z \quad (\text{in } (\mathbb{D}, \mathbb{D}, \| \cdot \|_\infty)) \]

where \(Z\) is a standard normally distributed random variable, \(f\) is the Lebesgue density of \(F\), \(c_\beta\) is some constant, and \(\mathbb{D}\) is the \(\sigma\)-algebra on \(\mathbb{D}\) generated by the usual coordinate projections; see e.g. [8, 15, 16]. Notice the asymptotic degeneracy of the limit process in (2) which shows that the increments of the standardized empirical distribution function \(\hat{F}_n\) over disjoint intervals, or disjoint observation sets, are asymptotically completely correlated. Also notice the noncentral rate \(\beta - 1/2\) in (2).

As indicated in the Introduction, for our purposes the use of the sup-norm \(\| \cdot \|_\infty\) in (2) is insufficient. We need a corresponding result for the weighted sup-norm \(\| \cdot \|_{\lambda} := \| \cdot \|_{\lambda, \infty}\). For \(\lambda \geq 0\), let \(\mathbb{D}_\lambda\) be the space of all càdlàg functions \(\psi\) on \(\mathbb{R}\) with \(\| \psi \|_{\lambda, \infty} < \infty\), and \(\mathbb{C}_\lambda\) be the subspace of all continuous functions in \(\mathbb{D}_\lambda\). We equip \(\mathbb{D}_\lambda\) with the \(\sigma\)-algebra \(\mathbb{D}_\lambda := \mathbb{D} \cap \mathbb{D}_\lambda\) to make it a measurable space, where as before \(\mathbb{D}\) is the \(\sigma\)-algebra generated by the usual coordinate projections. Without loss of generality we assume \(a_0 = 1\). The following theorem is proven in [6] using results from [1] and [23],

**Theorem 1** (NCLT for \(\hat{F}_n\)) Let \(\lambda \geq 0\), and assume that

(i) \(a_s = s^{-\beta} \ell(s), \ s \in \mathbb{N}\), where \(\beta \in (\frac{1}{2}, 1)\) and \(\ell\) is slowly varying at infinity,

(ii) \(\mathbb{E}[|\varepsilon_0|^{2+2\lambda}] < \infty\),

(iii) the distribution function \(G\) of \(\varepsilon_0\) is twice differentiable and \(\sum_{j=1}^{2} \int |G^{(j)}(x)|^2 \phi_{2\lambda}(x) dx < \infty\).

Then we have the following analogue of (2):

\[ n^{\beta-1/2} \ell(n)^{-1}(\hat{F}_n(\cdot) - F(\cdot)) \xrightarrow{d} c_{1,\beta} f(\cdot)Z \quad (\text{in } (\mathbb{D}_\lambda, \mathbb{D}_\lambda, \| \cdot \|_{\lambda, \infty})), \]

where \(f\) is the Lebesgue density of \(F\), \(Z\) is a standard normally distributed random variable, and \(c_{1,\beta} := (\mathbb{E}[\varepsilon_0^2](1 - (\beta - \frac{1}{2}))(1 - (2\beta - 1))/(\int_0^{\infty} (x + x^2)^{-\beta} dx))^{1/2}\).
NCLT for plug-in estimators based on long-memory sequences

We now turn to the application of the Modified FDM to $T(\hat{F}_n)$. First of all we recall from [4] the notion of quasi-Hadamard differentiability and the Modified FDM. Let $\mathbf{V}$ and $\mathbf{V}'$ be vector spaces, and $\mathbf{V}_0$ be a subspace of $\mathbf{V}$. Let $\| \cdot \|_{\mathbf{V}_0}$ and $\| \cdot \|_{\mathbf{V}'}$ be norms on $\mathbf{V}_0$ and $\mathbf{V}'$, respectively.

**Definition 2** (Quasi-Hadamard differentiability) Let $T : \mathbf{V}_T \to \mathbf{V}'$ be a mapping defined on a subset $\mathbf{V}_T$ of $\mathbf{V}$, and $\mathbb{C}_0$ be a subset of $\mathbf{V}_0$. Then $T$ is said to be quasi-Hadamard differentiable at $\theta \in \mathbf{V}_T$ tangentially to $\mathbb{C}_0(\mathbf{V}_0)$ if there is some continuous mapping $D_{\theta;\mathbb{C}_0(\mathbf{V}_0)}^\text{Had} T : \mathbb{C}_0 \to \mathbf{V}'$ such that

$$
\lim_{n \to \infty} \left\| D_{\theta;\mathbb{C}_0(\mathbf{V}_0)}^\text{Had} T(v) - \frac{T(\theta + h_nv_n) - T(\theta)}{h_n} \right\|_{\mathbf{V}'} = 0
$$

holds for each triplet $(v, (v_n), (h_n))$, with $v \in \mathbb{C}_0$, $(v_n) \subset \mathbf{V}_0$ satisfying $\|v_n - v\|_{\mathbf{V}_0} \to 0$ as well as $\theta + h_nv_n \in \mathbf{V}_T$ for every $n \in \mathbb{N}$, and $(h_n) \subset (0, \infty)$ satisfying $h_n \to 0$. In this case the mapping $D_{\theta;\mathbb{C}_0(\mathbf{V}_0)}^\text{Had} T$ is called quasi-Hadamard derivative of $T$ at $\theta$ tangentially to $\mathbb{C}_0(\mathbf{V}_0)$.

Let $\mathcal{V}_0$ and $\mathcal{V}'$ be $\sigma$-algebras on $\mathbf{V}_0$ and $\mathbf{V}'$, respectively. Suppose that $\mathcal{V}_0$ is nested between the open-ball and the Borel $\sigma$-algebra on $\mathbf{V}_0$, and that $\mathcal{V}'$ is not larger than the Borel $\sigma$-algebra on $\mathbf{V}'$. For every $n \in \mathbb{N}$, let $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ be a probability space, and $\hat{\theta}_n$ be a mapping from $\Omega_n$ to $\mathbf{V}$.

**Theorem 3** (Modified Functional Delta Method) Let $T : \mathbf{V}_T \to \mathbf{V}'$ be a mapping defined on some subset $\mathbf{V}_T$ of $\mathbf{V}$, let $\theta \in \mathbf{V}_T$, let $\mathbb{C}_0$ be some subset of $\mathbf{V}_0$ being separable w.r.t. $\| \cdot \|_{\mathbf{V}_0}$ (we regarded $\| \cdot \|_{\mathbf{V}_0}$ as a metric if $\mathbb{C}_0$ is not a vector space), and suppose that

(i) $\hat{\theta}_n$ takes values only in $\mathbf{V}_T$,

(ii) $\hat{\theta}_n - \theta$ takes values only in $\mathbf{V}_0$, is $(\mathcal{F}_n, \mathcal{V}_0)$-measurable and satisfies

$$
r_n(\hat{\theta}_n - \theta) \xrightarrow{d} \mathbf{V} \quad (\text{in } (\mathbf{V}_0, \mathbb{V}_0, \| \cdot \|_{\mathbf{V}_0}))
$$

for some sequence $(r_n) \subset (0, \infty)$ with $r_n \uparrow \infty$, and some random element $V$ of $(\mathbf{V}_0, \mathbb{V}_0)$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values only in $\mathbb{C}_0$,

(iii) $\tilde{\omega} \mapsto T(W(\tilde{\omega}) + \theta)$ is $(\tilde{\mathcal{F}}, \mathcal{V}')$-measurable whenever $W$ is a measurable mapping from some measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to $(\mathbf{V}_0, \mathbb{V}_0)$ such that $W(\tilde{\omega}) + \theta \in \mathbf{V}_T$ for all $\tilde{\omega} \in \tilde{\Omega}$,

(iv) $T$ is quasi-Hadamard differentiable at $\theta$ tangentially to $\mathbb{C}_0(\mathbf{V}_0)$ with quasi-Hadamard derivative $D_{\theta;\mathbb{C}_0(\mathbf{V}_0)}^\text{Had} T$.

Then

$$
r_n(T(\hat{\theta}_n) - T(\theta)) \xrightarrow{d} D_{\theta;\mathbb{C}_0(\mathbf{V}_0)}^\text{Had} T(V) \quad (\text{in } (\mathbf{V}', \mathcal{V}', \| \cdot \|_{\mathbf{V}'})�)
$$

As an immediate consequence of Theorems 1 and 3 we now obtain the following NCLT for the plug-in estimator $T(\hat{F}_n)$. We choose $\mathbf{V} := \mathbb{D}$, $\mathbf{V}_0 := \mathbb{D}_\lambda$, $\mathbb{C}_0 := \mathbb{C}_\lambda$, and assume that $\mathbf{V}_T$ is a class of distribution functions on the real line containing $F$.

**Theorem 4** (NCLT for $T(\hat{F}_n)$) Let $\lambda \geq 0$, and assume that

(i) $\hat{F}_n$ takes values only in $\mathbf{V}_T$,

(ii) the assumptions of Theorem 1 are fulfilled,

(iii) $\tilde{\omega} \mapsto T(W(\tilde{\omega}) + F)$ is $(\tilde{\mathcal{F}}, \mathcal{V}')$-measurable whenever $W$ is a measurable mapping from some measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to $(\mathbb{D}_\lambda, \mathbb{D}_\lambda)$ such that $W(\tilde{\omega}) + F \in \mathbf{V}_T$ for all $\tilde{\omega} \in \tilde{\Omega}$,
(iv) $T$ is quasi-Hadamard differentiable at $F$ tangentially to $C_{\lambda}(D_{\lambda})$ with quasi-Hadamard derivative $D_{F: C_{\lambda}(D_{\lambda})}^{\operatorname{Had}} T$.

Then
\[
\lim_{n \to \infty} n^{-1/2} \ell(n)^{-1} (T(F_n) - T(F)) = D_{F: C_{\lambda}(D_{\lambda})}^{\operatorname{Had}} T(c_{1,\beta} f(\cdot); Z) \quad \text{(in $(V', V', \|\cdot\|)$,)}
\]
where $\beta$, $c_{1,\beta}$, $f$ and $Z$ are as in Theorem 1.

**Example 5** (L-functionals) Let $K$ be the distribution function on $[0, 1]$, and $V_K$ be the class of all distribution functions $F$ on the real line for which $\int |x| dK(F(x)) < \infty$. The functional $L$, defined by
\[
L(F) := L_K(F) := \int x dK(F(x)), \quad F \in V_K,
\]
is called L-functional associated with $K$. It was shown in [4] that if $K$ is continuous and piecewise differentiable, the (piecewise) derivative $K'$ is bounded above and $F \in V_K$ takes the value $d \in (0, 1)$ at most once if $K$ is not differentiable at $d$, then for every $\lambda > 1$ the functional $L : V_K \to \mathbb{R}$ is quasi-Hadamard differentiable at $F$ tangentially to $C_{\lambda}(D_{\lambda})$ with quasi-Hadamard derivative
\[
D_{F: C_{\lambda}(D_{\lambda})}^{\operatorname{Had}} L(v) = \int K'(F(x)) v(x) dx \quad \forall v \in C_{\lambda}.
\]
Thus, if also the assumptions of Theorem 1 are fulfilled with $f \in C_{\lambda}$, Theorem 4 (with $V' = \mathbb{R}$) yields
\[
\lim_{n \to \infty} n^{-1/2} \ell(n)^{-1} \langle L(F_n) - L(F) \rangle = \tilde{Z} \quad \text{(in $(\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|$)},
\]
where $\tilde{Z}$ is normally distributed with mean zero and variance $c_{1,\beta}^2 (\int K'(F(x)) f(x) dx)^2$, and $\beta$ and $c_{1,\beta}$ are as in Theorem 1.

**Example 6** (V-functionals) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function, and $V_g$ be the class of all distribution functions $F$ on the real line for which $\int \int |g(x_1, x_2)| dF(x_1) dF(x_2) < \infty$. The functional $U$, defined by
\[
U(F) := U_g(F) := \int \int g(x_1, x_2) dF(x_1) dF(x_2), \quad F \in V_g,
\]
is called von Mises-functional (V-functional) associated with $g$. Let $\mathbb{B}V_{\operatorname{loc}}$ be the space of all functions $\psi : \mathbb{R} \to \mathbb{R}$ of local bounded variation. For $\psi \in \mathbb{B}V_{\operatorname{loc}}$, we denote by $d\psi^+$ and $d\psi^-$ the unique positive Radon measures induced by the Jordan decomposition of $\psi$, and we set $|d\psi| := d\psi^+ + d\psi^-$. Suppose that, for some $\lambda > \lambda' \geq 0$, the following two assertions hold:

(a) For every $x_2 \in \mathbb{R}$ fixed, the function $g_{x_2}(\cdot) := g(\cdot, x_2)$ lies in $\mathbb{B}V_{\operatorname{loc}} \cap \mathbb{D}_{-\lambda'}$. Moreover, the function $x_2 \mapsto \int \phi_{-\lambda}(x_1) g_{x_2}(x_1) dx_1$ lies in $\mathbb{D}_{-\lambda'}$.

(b) The functions $g_{1, F}(\cdot) := \int g(\cdot, x_2) dF(x_2)$ and $g_{2, F}(\cdot) := \int g(x_1, \cdot) dF(x_1)$ lie in $\mathbb{B}V_{\operatorname{loc}}$, and we have $\int \phi_{-\lambda}(x) |d\phi_{i, F}(x)| < \infty$ for $i = 1, 2$. Moreover, the functions $\overline{g_{1, F}}(\cdot) := \int |g(\cdot, x_2)| dF(x_2)$ and $\overline{g_{2, F}}(\cdot) := \int |g(x_1, \cdot)| dF(x_1)$ lie in $\mathbb{D}_{-\lambda'}$.

It is shown in [5] that under assumptions (a)–(b) the functional $U$ is quasi-Hadamard differentiable at $F$ tangentially to $C_{\lambda}(D_{\lambda})$ with quasi-Hadamard derivative
\[
D_{F: C_{\lambda}(D_{\lambda})}^{\operatorname{Had}} U(v) = -\int v(x) dg_{1, F}(x) - \int v(x) dg_{2, F}(x) \quad \forall v \in C_{\lambda}.
\]
Thus, if also the assumptions of Theorem 1 are fulfilled with $f \in C_{\lambda}$, Theorem 4 (with $V' = \mathbb{R}$) yields
\[
\lim_{n \to \infty} n^{-1/2} \ell(n)^{-1} \langle U(F_n) - U(F) \rangle = \tilde{Z} \quad \text{(in $(\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|$)},
\]
where \( \hat{Z} \) is normally distributed with mean zero and variance \( c_{1,\beta}^2(\int f(x)dg_{1,F}(x) + \int f(x)dg_{2,F}(x))^2 \), and \( \beta \) and \( c_{1,\beta} \) are as in Theorem 1.

It is easy to show that the variance kernel \( g(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2 \) and Gini’s mean difference kernel \( g(x_1, x_2) = |x_1 - x_2| \) satisfy conditions (a)–(b) for \( \lambda' = 2 \) and \( \lambda' = 1 \) (respectively), where \( dg_{1,F}(x) = dg_{2,F}(x) = (x - \mathbb{E}[X_1])dx \) and \( dg_{1,F}(x) = dg_{2,F}(x) = (2F(x) - 1)dx \) (respectively); cf. [5]. In the former case, however, it is straightforwardly seen that the asymptotic variance in (5) vanishes, so that the right-hand side in (5) degenerates to zero. This is consistent with Example 1 in [9].

\[ \diamond \]

**Remark 7 (Degenerate V-functionals)** Among V-functionals—introduced in Example 6—the functionals with a so-called degenerate kernel have attracted special interest; see, e.g., [8, 9]. A kernel \( g \) is called *degenerate* w.r.t. \( F \in \mathbf{V}_g \) if the functions \( g_{1,F} \) and \( g_{2,F} \) defined in part (b) in Example 6 are identically zero. In this case, \( \mathcal{U} \) is called *degenerate* V-functional w.r.t. \( F \). Moreover, in this case the right-hand side in (4) vanishes and thus the right-hand side in (5) degenerates to zero. That is, an application of Theorem 4 yields little. However, in this case one can exploit the Continuous Mapping Theorem (CMT) instead of the Modified FDM. Indeed: By the degeneracy of the kernel \( g \) we have the representation \( \mathcal{U}(\hat{F}_n) = \int \int g(x_1, x_2) d(\hat{F}_n - F)(x_1) d(\hat{F}_n - F)(x_2) \) and it was pointed out in [8, Section 2] that, under certain conditions on \( g \) and \( F \), integration-by-parts yields

\[ \mathcal{U}(\hat{F}_n) = \int \int (\hat{F}_n - F)(x_1)(\hat{F}_n - F)(x_2) \, dg(x_1, x_2). \]

To apply integration-by-parts, it was assumed in [8] that the kernel \( g \) is right-continuous and has bounded total variation. However, as the assumption that \( g \) be of bounded total variation is too restrictive, the result of [8, Section 2] was extended in [9] to more general kernels. A related, slightly stronger result can be found in [6]. Now, if the assumptions of Theorem 1 hold for some \( \lambda \geq 0 \) for which the integral \( \int g(x_1, x_2) \, dg(x_1, x_2) \) is finite, then we immediately obtain from (6), Theorem 1, \( \mathcal{U}(F) = 0 \) (which holds by the degeneracy of \( g \)) and the CMT that

\[
n^2 \beta^{-1} \ell(n)^{-2} \mathcal{U}(\hat{F}_n) \xrightarrow{d} c_{1,\beta}^2 \int \int f(x_1) f(x_2) \, dg(x_1, x_2) Z^2 \quad (\text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|)),
\]

where \( Z^2 \) is \( \chi^2 \)-distributed, and \( \beta \) and \( c_{1,\beta} \) are as in Theorem 1. For details, in particular for the conditions on \( g \) and \( F \) ensuring the representation (6), see [6].

\[ \diamond \]

**References**


Noncentral limit theorems for statistical functionals based on strictly stationary time series exhibiting long-range dependence are presented. The key tool is a noncentral limit theorem for empirical processes of long-memory data with respect to nonuniform sup-norms. Using a modified Functional Delta Method, based on the new concept of quasi-Hadamard differentiability, one can easily derive the asymptotic distribution of fairly general statistics, including L- and V-statistics.