Stationarity domains of Power threshold GARCH time series with heavy tailed generator process

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1. Introduction

Financial time series present a lot of features (stylised facts) reproduced by the several conditionally heteroscedastic models that, since Engle (1982), appeared in the literature.
A property of long memory in the shocks of the conditional variance was detected in this kind of data and led to the introduction of power GARCH models (Ding *et al* (1993) and Liu and Brorsen (1995)). The introduction of an exponent δ in the GARCH equation propagates not just conditional moments of order two but, more generally, absolute moments of order δ and, according with Ding *et al* (1993), the corresponding models incorporate the referred long memory property.
We consider here a natural extension of TGARCH processes that takes into account both long memory property and asymmetry in the stochastic volatility. This class of power-transformed and threshold GARCH (δ-TGARCH) models was introduced by Pan, Wang and Tong (2008) with a slightly different parameterization and they called them PTTGARCH models.
Let $X = (X_t,\ t \in \mathbb{Z})$ be a real stochastic process and, for any $t \in \mathbb{Z}$, let us consider $X_t^+ = X_t \mathbb{I}_{\{X_t \geq 0\}}$, $X_t^- = -X_t \mathbb{I}_{\{X_t < 0\}}$ and $\underline{X}_t$ the $\sigma-$ field generated by $(X_{t-i}, i \geq 0)$.
The stochastic process $X = (X_t,\ t \in \mathbb{Z})$ is said to follow a δ power threshold generalized autoregressive conditional heteroscedastic ($\delta-$TGARCH) model with orders p and q ($p, q \in \mathbb{N}$) if, for every $t \in \mathbb{Z}$, we have

$$\left\{ \begin{array}{l} X_t = \sigma_t \varepsilon_t \\ \sigma_t^\delta = \omega + \sum\limits_{i=1}^{p} \left[\alpha_i \left(X_{t-i}^+\right)^\delta + \beta_i \left(X_{t-i}^-\right)^\delta \right] + \sum\limits_{j=1}^{q} \gamma_j \sigma_{t-j}^\delta \end{array} \right. \tag{1.1}$$

for some constants $\delta > 0$, $\omega > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $i = 1, ..., p$, $\gamma_j \geq 0$, $j = 1, ..., q$, and where $\varepsilon = (\varepsilon_t,\ t \in \mathbb{Z})$ is a sequence of independent and identically distributed real random variables such that ε_t is independent of $\underline{X}_{t-1}$, for every $t \in \mathbb{Z}$. The process ε is called the generator process of X.
If $\gamma_j = 0$, $j = 1, ..., q$, the $\delta-$ TGARCH(p, q) model is simply denoted $\delta-$TARCH(p).

The probabilistic structure of these models is analyzed by discussing the strict and weak of order r stationarities ($r \in \mathbb{R}^+$) as well as the ergodicity. Considering particular δ-TGARCH models with Gaussian and non-Gaussian generator processes we discuss and compare the corresponding stationarity domains. As this kind of models is adequate to heavy tails data we compare these domains according to the weight of the tails of the distribution generator process.
In order to simplify the presentation we consider in the following $m = \max(p,q)$ and introduce $\alpha_i = \beta_i = 0$, $i = p+1,...,q$, if $q > p$, and $\gamma_j = 0, j = q+1,...,p$, if $q < p$. With this convention we have

$$\sigma_t^\delta = \omega + \sum_{i=1}^{m}\left[\alpha_i\left(X_{t-i}^+\right)^\delta + \beta_i\left(X_{t-i}^-\right)^\delta + \gamma_i\sigma_{t-i}^\delta\right].$$

To develop the probabilistic study of these models and following the idea present in Mittnik, Paolella and Rachev (2002), let us consider the vectorial stochastic process of $\mathbb{R}^m$, $Y = (Y_t, t \in \mathbb{Z})$, whose k-component, $Y_t^{(k)}$, has the following definition

$$\begin{cases} Y_t^{(1)} = \sigma_t^\delta \\ Y_t^{(k)} = \sum_{i=k}^{m}\left[\alpha_i\left(X_{t-i+k-1}^+\right)^\delta + \beta_i\left(X_{t-i+k-1}^-\right)^\delta + \gamma_i\sigma_{t-i+k-1}^\delta\right], \ k = 2,...,m. \end{cases}$$

This process satisfies the recurrence equation

$$Y_{t+1} = A_tY_t + B$$

where $(A_t,\ t \in \mathbb{Z})$ is a sequence of random square matrices of order m and B is a determinist vector of $\mathbb{R}^m$ given by

$$A_t = \begin{bmatrix} \left[\alpha_1\left(\varepsilon_t^+\right)^\delta + \beta_1\left(\varepsilon_t^-\right)^\delta + \gamma_1\right]e_1 + ... + \left[\alpha_{m-1}\left(\varepsilon_t^+\right)^\delta + \beta_{m-1}\left(\varepsilon_t^-\right)^\delta + \gamma_{m-1}\right]e_{m-1} & I_{m-1} \\ \alpha_m\left(\varepsilon_t^+\right)^\delta + \beta_m\left(\varepsilon_t^-\right)^\delta + \gamma_m & 0_{m-1} \end{bmatrix},$$

$$B = \begin{bmatrix} \omega e_1 \\ 0 \end{bmatrix},$$

with $e_1,...,e_{m-1}$ the canonical base of $\mathbb{R}^{m-1}$, I_{m-1} the identity matrix of $m-1$ order and 0_{m-1} the null vector of $\mathbb{R}^{m-1}$.
As we assume that $(\varepsilon_t,\ t \in \mathbb{Z})$ are independent and identically distributed random variables, the random matrices $(A_t,\ t \in \mathbb{Z})$ are also independent and identically distributed.

2. $\delta-$ TGARCH processes: strict stationarity and stationarity up to order δ

Let us consider any norm $\|.\|$ on the set $\mathbb{M}$ of the square matrices of order m and the following hypothesis on the matrices $(A_t,\ t \in \mathbb{Z})$:

(H1): The sequence $\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right)_{n\in\mathbb{N}}$ converges almost surely $(a.s.)$ to a strictly negative constant γ.

The existence of a stationary solution for the $\delta-$ TGARCH model is stated in Gonçalves, Leite and Mendes-Lopes (2011). In fact, if the sequence $\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right)_{n\in\mathbb{N}}$ satisfies the hypothesis **(H1)**, there exists a unique strictly stationary and ergodic solution, $(X_t,\ t \in \mathbb{Z})$, of the $\delta-$ TGARCH model (1.1). Moreover, if $E\left(\log^+\|A_0\|\right) < +\infty$, the hypothesis **(H1)** is necessary to the existence of a unique strictly stationary and ergodic solution X of that model.

When we are in presence of a $\delta-$TGARCH model of order m, $m \in \mathbb{N}$, with $\sigma_t^\delta = \omega + \alpha_m \left(X_{t-m}^+\right)^\delta + \beta_m \left(X_{t-m}^-\right)^\delta + \gamma_m \sigma_{t-m}^\delta$, the previous results can be expressed as follows.

Proposition. If $E\left\{\log\left[\alpha_m \left(\varepsilon_t^+\right)^\delta + \beta_m \left(\varepsilon_t^-\right)^\delta + \gamma_m\right]\right\}$ exists then a necessary and sufficient condition for the existence of a unique strictly stationary solution of the $\delta-$TGARCH model of order m where $\sigma_t^\delta = \omega + \alpha_m \left(X_{t-m}^+\right)^\delta + \beta_m \left(X_{t-m}^-\right)^\delta + \gamma_m \sigma_{t-m}^\delta$ is

$$E\left\{\log\left[\alpha_m \left(\varepsilon_t^+\right)^\delta + \beta_m \left(\varepsilon_t^-\right)^\delta + \gamma_m\right]\right\} < 0.$$

In order to analyze the existence of the order δ moment of the process X, solution of model (1.1), and its relationship with the stationarity, let us suppose

(H2): $E\left(|\varepsilon_t|^\delta\right) < +\infty$ and $P\left(\varepsilon_t = 0\right) \neq 1$.

We denote $E\left(|\varepsilon_t|^\delta\right) = \phi_\delta$, $E\left[\left(\varepsilon_t^+\right)^\delta\right] = \phi_{1,\delta}$ and $E\left[\left(\varepsilon_t^-\right)^\delta\right] = \phi_{2,\delta}$. A sufficient condition of strict stationarity as well as a necessary and sufficient condition of moments existence is presented in the following theorem.

Theorem. Under **(H2)**, $E\left(|X_t|^\delta\right)$ exists and is independent of t if and only if

$$\sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) < 1.$$

Moreover, the existence of $E\left(|X_t|^\delta\right)$ assures the strict stationarity of the process X satisfying the model (1.1).

Remarks.

1. As a consequence of the previous theorem and Holder inequality, we deduce that the stochastic process X is weak stationary up to the order δ ([1]) if and only if $\sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) < 1$.

2. If ε satisfies the hypothesis **(H2)** and $\sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) < 1$ then $E\left(|X_t|^\delta\right)$ exists, is independent of t and we have

$$E\left(|X_t|^\delta\right) = \frac{\omega \ \phi_\delta}{1 - \sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right)}.$$

From $X_t = \sigma_t \varepsilon_t$ we obtain, under the same condition, $E\left(\sigma_t^\delta\right) = \frac{\omega}{1 - \sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right)}$.

3. Stationarity domains for particular $\delta-$ TGARCH processes

In this section we analyse and compare the domains of strict stationarity and stationarity up to the order δ for $\delta-$TGARCH processes with particular and useful in practice generator processses. Namely,

[1] $(X_t,\ t \in \mathbb{Z})$ is weak stationary up to the δ order if all the joint moments of $(X_{t_1}, ..., X_{t_n})$ of order less or equal to δ exist and are equal to the corresponding joint moments of $(X_{t_1+h}, ..., X_{t_n+h})$, $h \in \mathbb{Z}$, that is,

$$E\left[(X_{t_1})^{j_1} ... (X_{t_n})^{j_n}\right] = E\left[(X_{t_1+h})^{j_1} ... (X_{t_n+h})^{j_n}\right]$$

with $j_1 \geq 0, ..., j_n \geq 0$, $j_1 + ... + j_n \leq \delta$, $(t_1, ...t_n) \in \mathbb{Z}^n, h \in \mathbb{Z}$.

we consider models with generator processes following laws of the generalized error distribution (GED) family, that is, with probability density function

$$f(x) = \frac{\nu \exp\left[-\frac{1}{2}\left|\frac{x}{\beta}\right|^{\nu}\right]}{\beta\Gamma\left(\frac{1}{\nu}\right)2^{\left(1+\frac{1}{\nu}\right)}}, \quad x \in \mathbb{R},$$

with $\beta = \left[\frac{2^{-\frac{2}{\nu}}\Gamma\left(\frac{1}{\nu}\right)}{\Gamma\left(\frac{3}{\nu}\right)}\right]^{\frac{1}{2}}$ and Γ the Gamma function (Domínguez-Molina *et al*).
Varying the shape parameter ν, we consider generator processes distributions with more or less heavy tails. In fact, if $\nu = 2$ we have a normal distribution while if $\nu < 2$ (resp., $\nu > 2$) we get more (resp., less) heavy tailed distributions. In particular, if $\nu = 1$ we obtain a Laplace distribution and we get the Uniform on $\left[-\sqrt{3}, \sqrt{3}\right]$ distribution when $\nu \to +\infty$. We point out that the variance exists for all the laws in the GED family and is equal to 1.
In this study we take also the case of a generator process without variance, namely with the standard Cauchy distribution. The corresponding strict stationarity domains are compared with those of the Gaussian case.
In the next examples we take the $\delta-$TARCH model with $\sigma_t^{\delta} = \omega + \alpha_m \left(X_{t-m}^{+}\right)^{\delta} + \beta_m \left(X_{t-m}^{-}\right)^{\delta}$, $m \in \mathbb{N}$, and consider several distributions for the generator process.

Example 1. Let us consider generator processes ε following laws of the GED family with shape parameter ν. A necessary and sufficient condition of existence of a strictly stationary solution X is

$$\frac{\log\left(\alpha_m \beta_m\right)}{\log\frac{\Gamma\left(\frac{1}{\nu}\right)}{\Gamma\left(\frac{3}{\nu}\right)} + \frac{2}{\nu}\Psi\left(\frac{1}{\nu}\right)} < \delta$$

where Ψ is the Euler psi-function, and a necessary and sufficient condition for the existence of $E\left(\left|X_t\right|^{\delta}\right)$ is

$$\frac{\left[\Gamma\left(\frac{1}{\nu}\right)\right]^{\frac{\delta}{2}-1}\Gamma\left(\frac{\delta+1}{\nu}\right)}{2\left[\Gamma\left(\frac{3}{\nu}\right)\right]^{\frac{\delta}{2}}}\left(\alpha_m + \beta_m\right) < 1.$$

The regions of strict stationarity of the X process for $\nu = 0.25, 1, 2$ and 100 are depicted in the Figure 1, where (x, y, z) corresponds to $(\alpha_m, \beta_m, \delta)$. We observe that strict stationarity domain becomes bigger while the tail of the generator process becomes heavier.

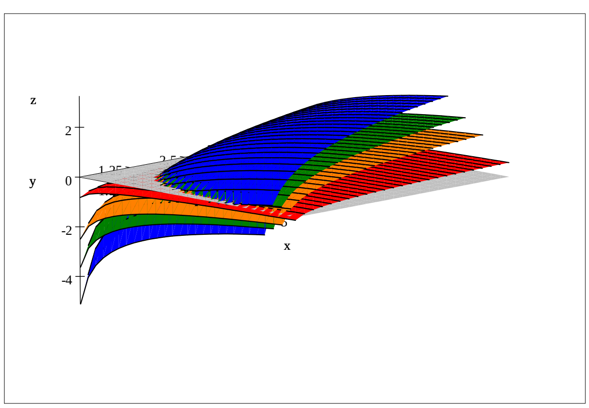

Fig. 1. Strict stationarity domains - GED (ν), $\nu = 0.25, 1, 2$ and 100
(the superficies bounding, at the right, the domains are graphed from down to up, respectively).

For the same models, the regions of up to the $\delta-$order weak stationarity are depicted in Figure 2.

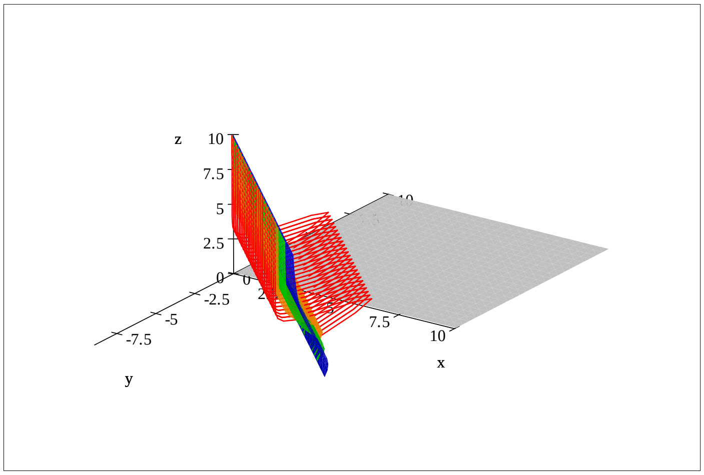

Fig. 2. Regions of up to $\delta-$ order weak stationarity - GED (ν), $\nu = 0.25, 1, 2$ and 100

In this case, we observe that for small values of δ the region of stationarity increases with the weight of the tails, while the situation reverses for large values of δ.

Example 2. If the generator process is Cauchy distributed, that is, with density $f(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$, we obtain

$$E\left\{\log\left[\alpha_m\left(\varepsilon_t^+\right)^\delta + \beta_m\left(\varepsilon_t^-\right)^\delta\right]\right\} = \tfrac{1}{2}\log\left(\alpha_m\beta_m\right).$$

A necessary and sufficient condition of strict stationarity for X is then $\alpha_m\beta_m < 1$, which is independent of the parameter δ. For $\delta < 1$, a necessary and sufficient condition for the existence of $E\left(|X_t|^\delta\right)$ is $(\alpha_m + \beta_m)\frac{1}{2\sin\left(\frac{\delta+1}{2}\pi\right)} < 1$ (Gonçalves, Leite and Mendes-Lopes, 2011).
The regions of strict and up to the $\delta-$order weak stationarity of the X process are depicted in Figure 3.

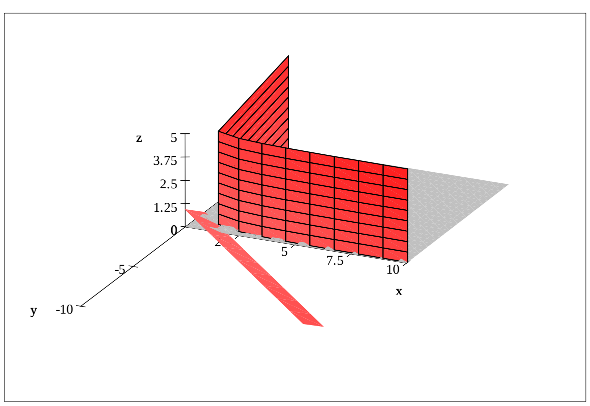

Fig. 3. Regions of strict and up to $\delta-$order weak stationarity - Cauchy generator process.

Finally, let us compare the strict stationarity domain of this example with that of a model with a Gaussian generator process. These domains are depicted in Figure 4.

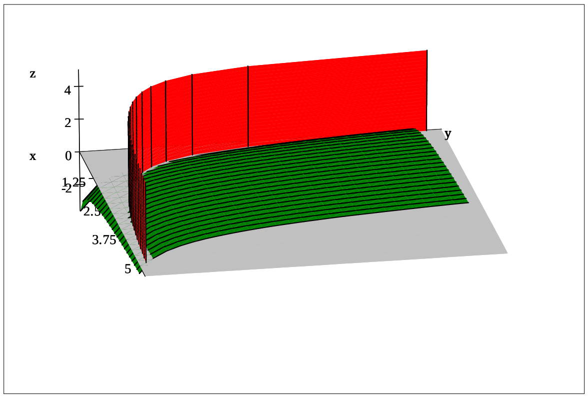

Fig. 4. Domains of strict stationarity - Cauchy and Gaussian generator processes.

Contrary to what we observe in the case of the GED family, the strict stationarity region decreases when the weight of the tails increases. We note that we are now comparing the Gaussian case with that corresponding to an heavy tailed generator process without variance while in the previous cases the comparison is made for generator processes with the same variance. Thus, the existence of the generator process variance seems to influence significantly the regions under study.

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