CIID frailty models and implied copulas

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Abstract

A unified stochastic framework for all portfolio default models with conditionally independent and identically distributed (CIID) default times is presented. Desirable statistical properties of dependent default times are introduced in an axiomatic manner and related to the unified framework. It is shown how commonly used models, stemming from quite different mathematical and economic motivations, can be translated into a multivariate frailty model. After a discussion of popular specifications in this regard, two new models are introduced. The vector of default times in the first approach has an Archimax survival copula. The second innovation is capable of producing default pattern with interesting statistical properties. The motivation for the latter approach is to add an additional source of jump frailty to a classical intensity-based approach. An approximation of the portfolio-loss distribution is available in both cases. The paper closes with a discussion of various generalizations of the generic framework.

Keywords: Portfolio credit derivative, De Finetti's Theorem, copula, large-homogeneous portfolio approximation, multivariate frailty model.

1 Introduction

Following the seminal work of [Vasicek 1987], various related portfolio default models have recently been proposed, see, e.g., [Li 2000, Frey, McNeil 2001, Schönbucher 2002, Hull, White 2004, Guégan, Houdain 2005, Baxter 2006, Kalemanova et al. 2007, Albrecher et al. 2007, Mai, Scherer 2009a] to provide some examples. Even though these papers use diverse economic motivations, rely on alternative mathematical techniques[1], and focus on different applications[2], all models share as common ground a large homogeneous portfolio approximation, providing a convenient tool for applications that require the loss distribution of some large portfolio[3]. In this paper, a unified stochastic framework for all models in this spirit is constructed. Such a treatment provides several advantages:

- First of all, the mathematical structure behind this class of models becomes fully transparent. Instead of relying on specific distributional assumptions and related mathematical concepts, we provide as a generic framework a multivariate frailty model that uses the classical theorems of de Finetti and Glivenko-Cantelli as key ingredients to obtain the portfolio-loss distribution. In contrast to several of the aforementioned examples, the present construction is consistent with respect to time and does not rely on some fix maturity (or a discrete number of maturities).

[1] The starting point might be a multivariate structural-default model, a certain dependence structure / copula for the vector of default times, a frailty model, or some latent-factor construction.
[2] The most important ones being risk-management and the pricing of portfolio derivatives.
[3] We focus on the pricing of insurance premium for tranches of a credit portfolio. Note, however, that applications to other insurance portfolios can be treated similarly.

1 Introduction

- Given a portfolio of d credits and denoting the vector of default times by $(\tau_1, \ldots, \tau_d)'$, various statistical properties have been investigated with respect to the implied dependence structure and resulting default pattern of the modelled default times, as well as with respect to the implied portfolio-loss distribution. Having a unified framework at hand, an objective comparison of alternative model specifications is facilitated. We axiomatically define a list of (desirable) statistical properties and investigate the unified framework in this regard. Obviously, this is much more efficient compared to a case-by-case analysis. Later on, we explicitly rewrite a battery of popular models in our language and investigate their statistical properties.
- Based on a generic framework, it is often easier to truly understand the mathematical concept behind a generalization of some model, e.g. to random recovery rates or hierarchical dependence structures. Hence, there is a fair chance that one can transfer the idea of a generalization from one class of models to some other. Moreover, for two concrete cases, we show how given models can be combined to a framework that inherits all desirable statistical features of the building blocks. We show that it is even possible to combine alternative models over time, using them as some sort of local correlation model. Finally, we obtain a deeper understanding of how far we can stretch the limits of CIID-models and, related, what model generalizations come at the price of losing the mathematical viability.

Throughout we consider a portfolio of d defaultable assets and let $(\tau_1, \ldots, \tau_d)'$ denote the vector of their default times. Both applications, the pricing of portfolio credit derivatives as well as risk management of credit portfolios, require the distribution of the accumulated loss within the reference portfolio up to time t. Currently, one of the most prominent applications in the context of portfolio credit derivatives is the pricing of collateralized debt obligations (CDOs). A CDO can be seen as an insurance contract for certain loss pieces of a credit portfolio. A convenient, and for sufficiently large portfolios widely used, assumption is a homogeneous portfolio structure with respect to recovery rates and portfolio weights. This allows to express the premium and default leg of the CDO's tranches as options on the (relative) portfolio-loss process $\{L_t\}_{t\geq 0}$, defined as $L_t := \frac{1}{d}\sum_{i=1}^{d} \mathbb{1}_{\{\tau_i \leq t\}}$, for $t \geq 0$. From a mathematical perspective, it is required to compute expectations of the form:

$$\mathbb{E}[f(L_t)] = \int_{[0,1]} f(x)\,\mathbb{P}(L_t \in dx), \quad f \text{ non-linear (collar type)},$$

where f depends on the considered tranche and the recovery rate. Hence, it is important to construct the vector of default times in such a way that the distribution of L_t can be identified or, at least, efficiently approximated. Due to the large dimensionality of the problem[4], one has to accept simplifying assumptions to circumvent time-consuming Monte-Carlo techniques. In this regard, a popular class of models is based on the following ansatz: there is a market factor M, conditioned on which all default times are iid with distribution function $t \mapsto F_t := \text{function}(M, t)$. The core motivation for these models is

[4]A typical convention for credit derivatives is $d = 125$, insurance portfolios are often even larger.

to approximate the distribution of L_t by the (more tractable) distribution of the market factor M. The seminal model in this spirit is [Vasicek 1987, Li 2000], specifying M as a normal random variable, which results in a Gaussian dependence structure. Since this copula has several drawbacks, e.g., zero tail dependence, symmetric dependence pattern, and an insufficient fit to quoted CDO spreads, several authors extended the approach to other market factors[5]. More dynamic models are obtained when the market factor $M = \{M_t\}_{t \geq 0}$ is a non-trivial stochastic process and $F_t := \text{function}(M_t)$. Such a model is proposed by [Mai, Scherer 2009a] with $\{M_t\}_{t \geq 0}$ being a Lévy subordinator[6].

From a practical perspective, a calibration of the model typically relies on market quotes of a) portfolio CDS and single-name CDS and b) CDO tranche spreads. Considering a), these are not affected by the dependence structure between the default times, but do depend on the respective univariate default probabilities. Hence, the required term-structures of univariate default probabilities can be extracted. Considering b), after having fixed the marginal default probabilities, spreads of the different tranches of a CDO can be used to calibrate the dependence parameters of the model. For this, it is very convenient if the model's dependence parameters do not affect the (already fixed) marginal default probabilities, i.e. the model allows for a separation of dependence structure from default probabilities. From a theoretical perspective, such a separation naturally invokes a copula model. When the model is to be estimated to observed losses, it is crucial to explicitly know the model's dependence structure. Thus, we are especially interested in models whose copula can be identified explicitly.

Besides the generic frailty model and the investigation of its statistical properties, we present as another contribution two new multivariate default models with very interesting statistical properties. Both models allow for a convenient approximation of L_t by the distribution of the market factor and can thus be implemented without Monte-Carlo simulation. The first ansatz is based on a scale mixture of Lévy processes. The resulting survival copula of $(\tau_1, \ldots, \tau_d)'$ is revealed as a scale mixture of Marshall-Olkin copulas, constituting a proper subclass of Archimax copulas, see e.g. [Li 2009, p. 253]. The second extension is based on processes of CGMY-type, see [Carr et al. 2003]. This model incorporates stylized facts such as default clusters and excess clustering. It

[5] For instance, [Hull, White 2004] use a Student t-distribution, [Guégan, Houdain 2005, Kalemanova et al. 2007] a NIG distribution, and [Albrecher et al. 2007] a general infinitely divisible distribution. In a related fashion, [Schönbucher 2002] assumes a positive random variable as market factor and constructs the model in such a way that the default times have an Archimedean survival copula. However, M is a single random variable in all aforementioned models, which equals the random parameter of a parametric family of distribution functions.

[6] The resulting survival copula of $(\tau_1, \ldots, \tau_d)'$ is of Marshall-Olkin kind, see [Mai, Scherer 2011]. The Marshall-Olkin distribution is well-studied and has several desirable properties for dependent defaults: an interpretation as a frailty model, asymmetric tail dependencies, and a singular component, i.e. positive joint default probabilities. Hence, Marshall-Olkin distributions have already been proposed for credit- and insurance-risk applications by [Giesecke 2003, Lindskog, McNeil 2003]. However, it is well-known that the Marshall-Olkin distribution is characterized by the lack-of-memory property, see e.g. [Marshall, Olkin 1967, Barlow, Proschan 1975, Galambos, Kotz 1978]. This implies a somewhat unrealistic assumption for dependent defaults, since it excludes direct contagion effects.

can be considered as an extension of a classical intensity-based ansatz in the spirit of [Duffie, Gârleanu 2001], when an additional source of frailty - a latent Lévy subordinator - is present.

The remaining paper is organized as follows: a general probabilistic framework for latent one-factor models and a review of commonly used examples (reformulated as frailty models) is given in Section 2. Two new models are introduced and discussed in Sections 3 and 4. Possible generalizations of the models are presented in Section 5. Besides technical proofs, the Appendix recalls, for the readers' convenience, the required notion of Lévy subordinators and the involved copula families.

2 A general CIID-framework

We consider a vector of default times $(\tau_1, \ldots, \tau_d)' \in [0, \infty)^d$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The first aim of this article is to present a generic representation that contains all aforementioned models and, in fact, all possible models relying on the assumption of conditionally independent and identically distributed (CIID) default times. Assume that $(\tau_1, \ldots, \tau_d)'$ is constructed on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by the following generic two-step method.

Definition 2.1 (The canonical CIID-construction)

1. *Let $\{F_t\}_{t \geq 0}$ be a non-decreasing, right-continuous stochastic process with left limits, such that $F_0 = 0$ and $\lim_{t \to \infty} F_t = 1$ hold almost surely. For fixed $\omega \in \Omega$, we consider $t \mapsto F_t(\omega)$ as the path of a distribution function of some random variable on $(0, \infty)$.*
2. *Conditioned on $\{F_t\}_{t \geq 0}$, let $(\tau_1, \ldots, \tau_d)'$ be iid with distribution function $t \mapsto F_t$.*

A canonical construction of such a model on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$\tau_k := \inf\left\{t \geq 0 : U_k \leq F_t\right\}, \quad k = 1, \ldots, d, \tag{1}$$

where $U_1, \ldots, U_d$ are iid with $U_k \sim \text{Uni}[0, 1]$ and $\{F_t\}_{t \geq 0}$ is independent of the vector $(U_1, \ldots, U_d)'$. Such a multivariate default model is called CIID-model in the following, CIID being the acronym of *conditionally independent and identically distributed.* On the one hand, this CIID-construction is a strong and restrictive assumption. For instance, it implies that the law of the default times is invariant under permutations of the components of $(\tau_1, \ldots, \tau_d)'$. In particular, each τ_k is distributed according to the distribution function $p(t) := \mathbb{E}[F_t]$, $t \geq 0$. Furthermore, it implicitly inherits a *large homogeneous portfolio assumption*, since the construction above is independent of the dimension d in the sense that one can consider (as an immediate extension of (1)) an infinite sequence $\{\tau_k\}_{k \in \mathbb{N}}$ of default times. On the other hand, a seminal theorem of De Finetti, see [De Finetti 1937], guarantees that *all* infinite exchangeable sequences of random variables can be constructed as above. This implies that the approach is more general than it might have appeared at first. From a practical perspective, the approach is general enough to comprise all commonly used default models which are tractable enough to circumvent a Monte-Carlo simulation when calibrated to market quotes. It is shown below how several popular models are embedded into the general CIID-framework from Definition 2.1 by identifying the respective specification of $\{F_t\}_{t\geq 0}$.

2.1 The portfolio loss distribution

The key advantage of CIID-models is that the distribution of the portfolio-loss process $L_t := \big(\mathbb{1}_{\{\tau_1\leq t\}} + \ldots + \mathbb{1}_{\{\tau_d\leq t\}}\big)/d$, $t\geq 0$, is available. More precisely:

$$\mathbb{P}\Big(L_t = \frac{k}{d}\Big) = \binom{d}{k}\,\mathbb{E}\big[F_t^k\,(1-F_t)^{d-k}\big],\quad k=0,1,\ldots,d.$$

For large $d \gg 2$ the complexity of the above expectation value as well as the size of the binomial coefficient prevent this formula from being of practical value. Since CIID-models are typically applied in large dimensions, the numerical difficulties are avoided by working with an infinite portfolio size (letting $d\to\infty$) which allows to approximate $\mathbb{P}(L_t\in dx)$ by $\mathbb{P}(F_t\in dx)$. For instance, it is not difficult to verify the following lemma, a proof of which is provided in the Appendix.

Lemma 2.2 (Approximation of the portfolio loss)
Consider the canonical probability space $(\Omega,\mathcal{F},\mathbb{P})$ *of a CIID-model as above. Then*

$$\mathbb{P}\Big(\lim_{d\to\infty}\sup_{t\geq 0}\big|F_t-L_t\big| = 0\Big) = 1.$$

Alternatively, for each $T>0$ *it holds true that*

$$\{L_t\}_{t\in[0,T]} \to \{F_t\}_{t\in[0,T]},\quad d\to\infty,$$

in the space $L^2(\Omega\times[0,T])$ *of square-integrable stochastic processes on* $[0,T]$.

As an application, the above result is used to justify approximations (for sufficiently large d) such as

$$\mathbb{E}\big[f(L_t)\big] = \int_{[0,1]} f(x)\,\mathbb{P}(L_t\in dx) \approx \int_{[0,1]} f(x)\,\mathbb{P}(F_t\in dx).$$

On a high level, approximation results such as Lemma 2.2 are called *large homogeneous portfolio approximation*. In our framework, it is possible to obtain this result as an application of the Theorem of Glivenko-Cantelli. Unlike most of the aforementioned references, we do not have to fix a certain time $t>0$. This is due to the new formulation as a frailty model, which reveals the underlying structure of (time consistent) CIID-models.

2.2 Properties of CIID-models

CIID-models in general are highly appreciated for their mathematical viability. However, for the selection of an appropriate model it is crucial to understand the different dependence structures that are implied by the various possible specifications. In an axiomatic way, a list of properties of the resulting vector of default times is specified below.

(Sep) The *separation* of dependence structure from marginals is extremely convenient for practical applications (e.g. the calibration or estimation of the model in two steps) and is also required for the derivation of the model's implied copula. Given the term structure of default probabilities, i.e. $t \mapsto p(t)$, the separation condition (Sep) is valid if the stochastic model for $\{F_t\}_{t\geq 0}$ is specified in such a way that $\mathbb{E}[F_t] = p(t)$ for all $t > 0$. This means that the randomness of $\{F_t\}_{t\geq 0}$ only affects the dependence structure, but not the marginal default probabilities.

(Cop) The joint distribution function of $(\tau_1, \ldots, \tau_d)'$ in a CIID-model, i.e. a model for the market frailty $\{F_t\}_{t\geq 0}$, is a priori implicit. More clearly, it is given by

$$\mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) = \mathbb{E}\big[\mathbb{E}\big[\prod_{k=1}^{d} \mathbb{1}_{\{\tau_k \leq t_k\}} \big| \{F_t\}_{t\geq 0}\big]\big] = \mathbb{E}\big[\prod_{k=1}^{d} \mathbb{E}\big[\mathbb{1}_{\{\tau_k \leq t_k\}} \big| \{F_t\}_{t\geq 0}\big]\big]$$
$$= \mathbb{E}\big[F_{t_1} \cdots F_{t_d}\big], \quad t_1, \ldots, t_d \geq 0.$$

In some specifications the latter expectation value can be computed explicitly and the multivariate distribution admits a well-known form. In such a case, one can conveniently rely on known statistical properties of the model to judge on its realism. Some distributions even allow for an intuitive economic interpretation. If, in addition, the model satisfies the separation property (Sep), then the marginal distributions of the default times are given a priori. In this case, the dependence structure can be studied from the *implied copula* or, if more convenient, from the *implied survival copula*. If the prespecified term structure of default probabilities $t \mapsto p(t)$ is continuous, then the implied copula C and the survival copula $\hat{C}$ of the default times are given by

$$C(u_1, \ldots, u_d) = \mathbb{E}\big[F_{p^{-1}(u_1)} \cdots F_{p^{-1}(u_d)}\big], \tag{2}$$
$$\hat{C}(u_1, \ldots, u_d) = \mathbb{E}\big[(1 - F_{p^{-1}(1-u_1)}) \cdots (1 - F_{p^{-1}(1-u_d)})\big], \tag{3}$$

where $p^{-1}(\cdot)$ denotes the generalized inverse of $p(\cdot)$ and $u_1, \ldots, u_d \in [0, 1]$.

(Exc) Time series of realized corporate defaults or insurance claims often exhibit points in time with accumulations of defaults. This property is termed *excess clustering.* It might even be reasonable to support *multiple defaults* at the same time. In the general CIID-framework this corresponds to possible jumps in the paths of $\{F_t\}_{t\geq 0}$. In the language of multivariate distribution functions, this corresponds to a singular component of the implied copula of $(\tau_1, \ldots, \tau_d)'$.

2.2 Properties of CIID-models

(Fs) The *qualitative structure* of the underlying *frailty distribution* $\{F_t\}_{t\geq 0}$ is important to understand the dynamics of the model. Three cases are distinguished:

($\text{Fs}_\ominus$) The source of frailty is static, i.e. for each $t > 0$, F_t is measurable with respect to the σ-algebra $\bigcap_{u>0} \sigma(F_s : 0 \leq s \leq u)$. This situation is typical for models which define $\{F_t\}_{t\geq 0}$ as a member of a parametric family of distribution functions with randomly drawn parameter. In most cases, $\{F_t\}_{t\geq 0}$ is monotonically affected by this parameter. This prevents the model from supporting changing market conditions, since the market frailty process $\{F_t\}_{t\geq 0}$ cannot change randomly.

($\text{Fs}_\odot$) The source of frailty is dynamic, but the innovations of the process $\{F_t\}_{t\geq 0}$ are driven by a time-homogeneous stochastic process. Interpreted from an economic perspective, this implies that the market uncertainty is affected by random changes, but these changes occur in a time-homogeneous pattern.

($\text{Fs}_\oplus$) The source of frailty is dynamic and the innovations of the process $\{F_t\}_{t\geq 0}$ are driven by a time-inhomogeneous stochastic process. From the economic point of view, this allows for realizations with randomly varying default environments. In particular, a typical realization of $\{F_t\}_{t\geq 0}$ inherits time periods with different local default probabilities and dependence structures.

(Tdc) A measure of dependence for the likelihood of joint early defaults is *tail dependence*. In the context of default risk, a positive lower tail dependence coefficient of the default times corresponds to a positive limit (as time goes to zero) of pairwise default correlations, see [Schönbucher 2003, Chapter 10]. Hence, this property is of specific interest for models with very small default probabilities or small time horizon. Additionally, empirical studies suggest that models supporting positive lower tail dependence of the default times are more successful in explaining CDO quotes. In mathematical terms, the lower tail dependence coefficient λ_l of a pair $(\tau_i, \tau_j)'$ of default times in a CIID-setup is given by[7]

$$\lambda_l := \lim_{t\downarrow 0} \mathbb{P}(\tau_i \leq t \,|\, \tau_j \leq t) = \lim_{t\downarrow 0} \frac{\mathbb{E}\big[F_t^2\big]}{\mathbb{E}[F_t]}. \tag{4}$$

(Den) When implementing the model, it is convenient if the distribution of F_t is tractable for all $t > 0$. Most convenient is the case when the *density* of F_t is available. Some specifications allow for a closed-form expression without special functions. In some models the density is only available through Laplace-inversion techniques.

It is shown below how various popular models are comprised in our general CIID-framework. These models are then discussed with regard to the aforementioned properties.

[7]Thus, this important measure of dependence is related to the specification of F_t in a rather simple way. The derivation of (4) is straightforward and therefore omitted.

2.3 The Gaussian copula model and extensions

[Vasicek 1987, Li 2000] generalize the seminal univariate structural default model of [Merton 1974] to d identical firms. Dependence is introduced through a single-factor structure: the idiosyncratic factors $\epsilon_1, \ldots, \epsilon_d$ and the market factor M are iid standard normally distributed random variables. Given the prespecified term structure of default probabilities $t \mapsto p(t)$ as model input, the default time of firm k is defined as

$$\begin{aligned}\tau_k &:= \inf\big\{t \geq 0 : \sqrt{\rho}\, M + \sqrt{1-\rho}\, \epsilon_k \leq \Phi^{-1}\big(p(t)\big)\big\} \qquad (5)\\ &= \inf\Big\{t \geq 0 : U_k \leq \Phi\Big(\frac{\Phi^{-1}\big(p(t)\big) - \sqrt{\rho}\, M}{\sqrt{1-\rho}}\Big)\Big\}, \quad k = 1, \ldots, d,\end{aligned}$$

where $\rho \in (0,1)$ adjusts the dependence, Φ denotes the distribution function of the standard normal law, and $U_1, \ldots, U_d$ are iid and obtained by $U_k := \Phi(\epsilon_k) \sim \text{Uni}[0,1]$. By construction, the vector $(\tau_1, \ldots, \tau_d)'$ has a Gaussian copula with identical pairwise correlation ρ as dependence structure and marginal distributions $\mathbb{P}(\tau_k \leq t) = p(t)$. Reformulating the model in the general CIID-setting one obtains $F_t := \Phi\big(\big(\Phi^{-1}\big(p(t)\big) - \sqrt{\rho}\, M\big)/\sqrt{1-\rho}\big)$, for $t \geq 0$. Summing up, the distribution of L_t can be approximated via the standard normal distribution of M. Generalizing this approach to distributions other than the normal, [Hull, White 2004] propose to replace it by the (heavier tailed) Student t-distribution. In a similar spirit, [Albrecher et al. 2007] consider a Lévy process $X = \{X_t\}_{t \in [0,1]}$, satisfying $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = 1$. Letting $X^{(0)}, \ldots, X^{(d)}$ be $d+1$ independent copies of X and $\rho \in (0,1)$, they define the individual factors $\epsilon_k := X^{(k)}_{1-\rho}$, $k = 1, \ldots, d$, and the market factor $M := X^{(0)}_{\rho}$. Then, construction (5) is replaced by

$$\begin{aligned}\tau_k &:= \inf\big\{t \geq 0 : M + \epsilon_k \leq H^{-1}_{[1]}\big(p(t)\big)\big\}\\ &= \inf\big\{t \geq 0 : U_k \leq H_{[1-\rho]}\big(H^{-1}_{[1]}\big(p(t)\big) - M\big)\big\}, \quad k = 1, \ldots, d,\end{aligned}$$

where $H_{[t]}$ denotes the distribution function of X_t and $U_1, \ldots, U_d$ are iid and obtained by $U_k := H_{[1-\rho]}(\epsilon_k) \sim \text{Uni}[0,1]$. This obviously corresponds to the choice $F_t := H_{[1-\rho]}\big(H^{-1}_{[1]}\big(p(t)\big) - M\big)$, for $t \geq 0$, in our general CIID-setup. When X is a Brownian motion, this approach is equivalent to (5). Considering other specifications, [Moosbrucker 2006] uses a Variance-Gamma process, [Guégan, Houdain 2005, Kalemanova et al. 2007] a Normal Inverse Gaussian (NIG) process, and [Baxter 2006] the sum of a Brownian motion and a Variance-Gamma process. Again, the distribution of L_t can be approximated via the distribution of the single random variable M, which is easy to handle.

Properties of the model

(Sep) In this specification one can take $t \mapsto p(t)$ as model input for the univariate marginal laws and obtains $\mathbb{E}[F_t] = p(t)$, $t \geq 0$.

(Cop) The copula behind the model is identified in the Gaussian specification and for Student t-factors as the respective distribution's copula. For the general Lévy framework, the implicitly defined copula in (2) and (3) is not well-studied.

(Exc) In all specifications, the resulting copula (explicitly or implicitly given) does not have a singular component, so multiple defaults at the same time are impossible. Equivalently, $t \mapsto F_t$ is almost surely continuous (for continuous $t \mapsto p(t)$).

(Fs$_\ominus$) The randomness in this class of models is induced by the randomness of the parameter of an otherwise deterministic distribution function. This makes these models static and also difficult to interpret.

(Tdc) One major disadvantage of a Gaussian dependence structure is zero tail dependence, see [McNeil et al. 2005, p. 211]. This means that joint early defaults are very unlikely. For other model specifications with a heavier tailed common factor, positive tail dependence is possible. For instance, it is shown in [Albrecher et al. 2007] that for the Lévy construction, the lower tail dependence coefficient of any pair of default times is given by

$$\lambda_l = \lim_{x \to -\infty} \int_{\mathbb{R}} \frac{H_{[1-\rho]}(x-y)^2}{H_{[1]}(x)} dH_{[\rho]}(y).$$

(Den) The density of F_t, $t > 0$, is known for various choices of the common factor, making this class of models quite viable.

When calibrating the Gaussian model to the CDO market it is often the case that for matching spreads of senior tranches extremely high correlation parameters are required. When distributions with heavier tails are used, e.g. the NIG model, the model seems to be better suited for a calibration to the CDO market.

2.4 A model based on mixtures of exponential distributions

[Marshall, Olkin 1988] show that the dependence structure behind iid exponential random variables with randomly drawn parameter is Archimedean. Denoting the Laplace transform of the positive random variable M by $\varphi(x) := \mathbb{E}[\exp(-x\,M)]$, $x \geq 0$, it follows that $\mathbb{P}\big(\epsilon_1/M > t_1, \ldots, \epsilon_d/M > t_d\big) = C_\varphi\big(\varphi(t_1), \ldots, \varphi(t_d)\big)$, $t_1, \ldots, t_d \geq 0$, where $\epsilon_1, \ldots, \epsilon_d$ are iid unit exponentially distributed and independent of M. The function $C_\varphi(u_1, \ldots, u_d) := \varphi\big(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_d)\big)$ is called *Archimedean copula* with *generator* φ. Transforming the components to standard uniform marginals, it follows that

$$(V_1, \ldots, V_d)' := \Big(\varphi\Big(\frac{\epsilon_1}{M}\Big), \ldots, \varphi\Big(\frac{\epsilon_d}{M}\Big)\Big)' \sim C_\varphi.$$

[Schönbucher 2002] uses this probabilistic model for portfolio credit risk and derives a large homogeneous portfolio approximation. Formulated as a frailty model and using the notations from above, given the prespecified term structure of default probabilities $t \mapsto p(t)$, one defines[8]

$$\begin{aligned}\tau_k &:= \inf\big\{t \geq 0 : 1 - p(t) \leq V_k\big\},\\ &= \inf\big\{t \geq 0 : U_k \leq 1 - \exp\big(-M\varphi^{-1}(1-p(t))\big)\big\}, \quad k = 1,\dots,d,\end{aligned}$$

where $U_1,\dots,U_d$ are iid obtained by $U_k := 1 - \exp(-\epsilon_k) \sim \text{Uni}[0,1]$. Translating this construction into the present CIID-setup yields $F_t := 1 - \exp\big(-M\,\varphi^{-1}\big(1-p(t)\big)\big)$, for $t \geq 0$. Summarizing, this implies that the distribution of L_t can be approximated using the distribution of M and default times defined in this way have an Archimedean survival copula C_φ[9].

Properties of the model

(Sep) The separation property holds, i.e. the term structure of default probabilities $t \mapsto p(t)$ can be prespecified and $\mathbb{E}[F_t] = 1 - \varphi\big(\varphi^{-1}(1-p(t))\big) = p(t)$, $t \geq 0$.

(Cop) By construction, the copula behind the model is of Archimedean kind. Such copulas, being parameterized by a function φ, are quite flexible. On the other side, they do not provide a firm economic interpretation.

(Exc) Multiple defaults at the same time are not possible, since Archimedean copulas do not assign positive mass to the diagonal of the unit cube. Stated differently, the process $\{F_t\}_{t\geq 0}$ is almost surely continuous (for continuous $t \mapsto p(t)$).

($\text{Fs}_\ominus$) As outlined above, the model is based on an exponential distribution with randomly chosen parameter. Hence, the model is static and difficult to interpret from an economic perspective.

(Tdc) The upper and lower tail dependence parameters of the Archimedean copula C_φ are given by

$$\lambda_u = \left\{\begin{array}{ll} 0, & \varphi'(0) < \infty \\ 2 - 2\lim\limits_{t\downarrow 0}\frac{\varphi'(2\,t)}{\varphi'(t)}, & \text{else}\end{array}\right\}, \quad \lambda_l = 2\lim_{t\uparrow\infty}\frac{\varphi'(2\,t)}{\varphi'(t)},$$

see [Joe 1997, p. 103ff]. For several classes, these parameters are positive and might even be unequal.

[8] The first line indicates the idea of [Schönbucher 2002]: starting from the canonical construction of a default time with distribution function $t \mapsto p(t)$, see [Schönbucher 2003, p. 122], dependent trigger variables $(V_1,\dots,V_d)'$ are used as the source of dependence.

[9] If one wishes to define default times in such a way that they have C_φ as copula instead of survival copula, one must use $F_t := \exp\big(-M\,\varphi^{-1}\big(p(t)\big)\big)$, $t \geq 0$. This can be deduced by replacing $(V_1,\dots,V_d)'$ in the above derivation by $(1-V_1,\dots,1-V_d)'$. This alternative ansatz can be used to switch tail dependencies: the lower tail dependence of C_φ equals the upper tail dependence of its survival copula, and vice versa. Since lower tail dependence between default times is desirable, one should use the latter approach when C_φ exhibits lower tail dependence and the first approach when C_φ exhibits upper tail dependence.

(Den) The density of F_t, $t > 0$, is known for various choices of the common factor, rendering this class of models quite viable. A list of popular generator functions φ and their associated random variables M is provided in, e.g., [Charpentier, Segers 2009].

2.5 An intensity-based approach

On a univariate level, intensity-based models are introduced and developed further in, e.g., [Jarrow, Turnbull 1995, Madan, Unal 1998, Lando 1998, Duffie, Singleton 1999]. On a multivariate level, so called *doubly-stochastic approaches* and extensions thereof are considered in, e.g., [Duffie, Gârleanu 2001, Das et al. 2007, Duffie et al. 2007, Yu 2007]. A single-factor specification, which fits into the setup of general CIID-models, can be constructed as a special case of the model in [Duffie, Gârleanu 2001]. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider a positive stochastic process $\{\lambda_t\}_{t\geq 0}$, which is $\mathbb{P}$-almost surely integrable on $[0, t]$ for all $t > 0$ and satisfies $\int_{(0,\infty)} \lambda_s\, ds = \infty$. Independently of this *market intensity* process, let $\epsilon_1, \ldots, \epsilon_d$ be iid unit exponential random variables. The vector $(\tau_1, \ldots, \tau_d)'$ of default times is defined by setting

$$\tau_k := \inf\big\{t > 0 \,:\, M_t \geq \epsilon_k\big\}, \quad M_t := \int_0^t \lambda_s\, ds, \quad k = 1, \ldots, d.$$

Translated into the setup of CIID-models, this is equivalent to modeling $\{F_t\}_{t\geq 0}$ as $F_t := 1 - \exp(-M_t)$, for $t \geq 0$. A prominent choice for $\{\lambda_t\}_{t\geq 0}$ is a *basic affine process*. This means that $\{\lambda_t\}_{t\geq 0}$ has parameters $(\kappa, \theta, \sigma, \mu, l)$ and is defined as the (unique) solution of the stochastic differential equation (SDE)

$$d\lambda_t = \kappa\,(\theta - \lambda_t)\, dt + \sigma\,\sqrt{\lambda_t}\, dB_t + dZ_t, \quad \lambda_0 > 0, \tag{6}$$

where $\{B_t\}_{t\geq 0}$ is a standard Brownian motion and $\{Z_t\}_{t\geq 0}$ is an independent compound Poisson process with intensity l and exponential jump sizes with mean $1/\mu$. Besides the immediate interpretation of the SDE for λ, one important advantage of using basic affine processes is that the Laplace transform of M_t is available from general results in [Duffie et al. 2000]. More clearly, it is known that

$$\mathbb{E}\big[e^{-x\, M_t}\big] = e^{\alpha(x,t)+\beta(x,t)\,\lambda_0}, \quad x \geq 0, \tag{7}$$

where the functions α and β are given by

$$\beta(x,t) = \frac{1 - e^{b(x)\,t}}{c(x) + d(x)\, e^{b(x)\,t}}, \tag{8}$$

$$\begin{aligned} \alpha(x,t) = 2\kappa\theta\Big(&-\sigma^{-2}\log\big(\frac{c(x)+d(x)e^{b(x)t}}{c(x)+d(x)}\big) + \frac{t}{2c(x)}\Big) \\ &+ l\Big(\frac{2\mu\log\big(1 - e^{b(x)t} - \mu(c(x)+d(x)e^{b(x)t})\big)}{2\mu\kappa + 2x - \sigma^2\mu^2}\Big) \\ &+ l\Big(\frac{-t}{1-\mu c(x)} - \frac{2\mu\log\big(-\mu(c(x)+d(x))\big)}{2\mu\kappa + 2x - \sigma^2\mu^2}\Big), \end{aligned} \tag{9}$$

with $b(x)$, $c(x)$, and $d(x)$ defined by

$$b(x) = -\sqrt{\kappa^2 + 2\,\sigma^2\,x}, \quad c(x) = \frac{\kappa + \sqrt{\kappa^2 + 2\,\sigma^2\,x}}{-2\,x}, \quad d(x) = \frac{-\kappa + \sqrt{\kappa^2 + 2\,\sigma^2\,x}}{-2\,x}.$$

This allows to compute $p(t) = \mathbb{E}[F_t]$ in closed form, as $p(t) = 1 - \mathbb{E}\big[\exp(-M_t)\big] = 1 - \exp\big(\alpha(1,t) + \beta(1,t)\,\lambda_0\big)$.

Properties of the model

(Sep) All parameters $(\kappa, \theta, \sigma, \mu, l)$ of a basic affine process enter the formulas for the marginal and the joint default probabilities. There is no parameter that solely affects the dependence structure; a separation is not possible. This complicates the calibration of the model and the interpretation of the parameters. However, the five parameters $(\kappa, \theta, \sigma, \mu, l)$ provide a good fit of the function $t \mapsto p(t) = \mathbb{E}[F_t]$ to market quotes of single-name CDS or portfolio CDS.

(Cop) The copula behind the model is unknown. This makes it difficult to study the underlying dependence structure of default times.

(Exc) [Das et al. 2007] find evidence that intensity-based approaches fail to explain excess clustering as observed in the markets, e.g. during the recent credit crisis. This is due to the fact that the integrated intensity process is continuous, and, hence, the random distribution function $\{F_t\}_{t\geq 0}$ is continuous, too. In Section 4 we propose a new extension of this intensity-based approach to incorporate excess clustering. This is achieved by incorporating jumps into $\{F_t\}_{t\geq 0}$.

($\text{Fs}_{\oplus}$) The intensity process $\{\lambda_t\}_{t\geq 0}$ is interpreted as an instantaneous default rate, making the model quite intuitive. The larger the intensity λ_t, the larger is the default probability over $[t, t+dt]$. Consequently, a typical realization of $\{F_t\}_{t\geq 0}$ inherits time periods with different local default probabilities, resulting from periods with high or low λ_t.

(Tdc) For a specification of the model using a basic affine process $\{\lambda_t\}_{t\geq 0}$, the resulting bivariate lower tail dependence coefficient (4) is zero. The required computation is very tedious and postponed to the Appendix.

(Den) For the approximation $\mathbb{P}(L_t \in dx) \approx \mathbb{P}(F_t \in dx)$ the density of M_t is required, which in the basic affine case can be obtained from the known Laplace transform via Laplace inversion. Even though this makes the implementation of the model more involved, it is still more efficient compared to a Monte-Carlo simulation. References for Laplace inversion algorithms, based on different theoretical inversion formulas, include [Talbot 1979, Abate et al. 1996, Abate, Valkó 2002, Abate, Valkó 2004]. We performed several numerical experiments and identified Talbot-type algorithms to be best suited for the present problem.

2.6 A model based on Lévy subordinators

The model of [Mai, Scherer 2009a] is related to the intensity-based approach, the key difference being that the (continuous) integrated intensity process is replaced by a (discontinuous) jump process. To set up the model, let $\epsilon_1, \ldots, \epsilon_d$ be iid random variables with unit exponential distribution and let $0 \neq \Lambda = \{\Lambda_t\}_{t \geq 0}$ be an independent (killed) Lévy subordinator with Laplace exponent Ψ. A brief introduction to Lévy subordinators is provided in the Appendix. Given the prespecified continuous and strictly increasing term structure of default probabilities $t \mapsto p(t)$, one defines the cumulated hazard function $t \mapsto h(t) := -\log\big(1 - p(t)\big)$, and sets the market frailty process as $M_t := \Lambda_{h(t)/\Psi(1)}$, $t \geq 0$. The default times are defined as

$$\tau_k := \inf\big\{t \geq 0 : M_t \geq \epsilon_k\big\} = \inf\big\{t \geq 0 : U_k \leq 1 - e^{-M_t}\big\}, \quad k = 1, \ldots, d,$$

where $U_1, \ldots, U_d$ are iid obtained by $U_k := 1 - \exp(-\epsilon_k) \sim \text{Uni}[0,1]$. Note that the expectation of Λ, evaluated at $h(t)/\Psi(1)$, is required to match the prespecified term structure of default probabilities $t \mapsto p(t)$, i.e. $\mathbb{E}[F_t] = p(t)$. Translating it into the framework of a general CIID-model, this approach corresponds to defining $F_t := 1 - \exp(-M_t)$, where $M_t := \Lambda_{h(t)/\Psi(1)}$ for $t \geq 0$.

Properties of the model

(Sep) A separation of marginals from dependence structure is valid: independently of the choice of subordinator Λ and with prespecified term structure $t \mapsto p(t)$, it holds that $\mathbb{E}[F_t] = 1 - \exp\big(-h(t)\big) = p(t)$, $t \geq 0$.

(Cop) The survival copula of the default times is known to be of Marshall-Olkin kind, see [Mai, Scherer 2011]. Note that the Marshall-Olkin distribution has already been proposed for credit-risk modeling by [Giesecke 2003, Lindskog, McNeil 2003], since it provides an intuitive interpretation as an exogenous shock model.

(Exc) Multiple defaults at the same time are possible, since the subordinator Λ can jump across more than one trigger variable at a time. Hence, the survival copula behind the multivariate default model (which is an exchangeable Marshall-Olkin copula) has a singular component on the diagonal and the model supports joint defaults. This property distinguishes the model from all aforementioned model specifications.

($\text{Fs}_\odot$) Since the common factor is a stochastic process instead of a single random variable, one obtains a dynamic structure of $\{F_t\}_{t\geq 0}$, and, hence, of the loss process $\{L_t\}_{t \geq 0}$ as well. For instance, it is possible that in some time periods default probabilities jump rapidly and in other time periods rather slowly. Unfortunately, the dependence structure behind the default times exhibits the so-called *multivariate lack-of memory property*, see e.g. [Marshall, Olkin 1967, Galambos, Kotz 1978]. Heuristically, the Lévy properties of Λ, corresponding to the lack-of memory property of the Marshall-Olkin distribution, force jumps of the market frailty to occur in time-homogeneous pattern.

(Tdc) The lower tail dependence coefficient of any pair of default times equals $\lambda_l = 2 - \Psi(2)/\Psi(1)$, see [Mai, Scherer 2009a], which is always positive unless $\Lambda_t = t$, $t \geq 0$.

(Den) Several classes of Lévy subordinators have known densities. Examples include the Inverse Gaussian and the Gamma subordinator. Other examples have semi-explicit densities, e.g., the stable subordinator and several compound Poisson subordinators[10].

In the following sections, two extensions of this modeling approach are presented. These generalizations aim at combining the desirable properties of the aforementioned models while preserving their viability. The first generalization combines the models of [Schönbucher 2002] and [Mai, Scherer 2009a]. The implied dependence structure is of Archimax kind. The second generalization combines the intensity-based approach with the model of [Mai, Scherer 2009a]. The result is a so-called *triply-stochastic model* which supports default clustering.

3 A new model based on scale mixtures of Marshall-Olkin copulas

It is possible to unify the approaches of [Schönbucher 2002] and [Mai, Scherer 2009a] into a framework generalizing both. In the language of copula theory, this corresponds to combining Archimedean with Marshall-Olkin copulas, which represents a family of copulas termed *scale mixtures of Marshall-Olkin copulas*, see [Li 2009]. It constitutes a proper subclass of so-called Archimax copulas, as introduced for the bivariate case in [Capéraà et al. 2000]. Let $\bar{M} > 0$ be a random variable with Laplace transform $\varphi(x) := \mathbb{E}[\exp(-x\,\bar{M})]$, $x \geq 0$, and let $\Lambda \neq 0$ be an independent (killed) Lévy subordinator with Laplace exponent Ψ. Independently of $(\bar{M}, \Lambda)$, let $\epsilon_1, \ldots, \epsilon_d$ be iid unit exponentially distributed. Given the continuous and strictly increasing prespecified term structure of default probabilities $t \mapsto p(t)$, one defines the market frailty process $M_t := \Lambda_{\bar{M}\varphi^{-1}(1-p(t))/\Psi(1)}$. The default times are then defined by

$$\tau_k := \inf\{t \geq 0 : M_t \geq \epsilon_k\} = \inf\{t \geq 0 : U_k \leq 1 - e^{-M_t}\}, \quad k = 1, \ldots, d,$$

where $U_1, \ldots, U_d$ are iid obtained by $U_k := 1 - \exp(-\epsilon_k) \sim \text{Uni}[0,1]$. In particular, choosing the calendar time $\Lambda_t = t$, $t \geq 0$, the model is equivalent to that of [Schönbucher 2002].

[10] This construction contains the simple Marshall-Olkin model with one armageddon shock of [Burtschell et al. 2009] as a special case. Rewriting this example in the present framework, the subordinator Λ must be linearly increasing until a single jump to infinity simultaneously destroys all components.

Similarly, the choice $\bar{M} \equiv 1$ implies the model of [Mai, Scherer 2009a]. Rewriting the model in the general CIID-setup, this means that

$$F_t := 1 - e^{-M_t}, \quad M_t := \Lambda_{\bar{M}\varphi^{-1}(1-p(t))/\Psi(1)}, \quad t \geq 0.$$

Suitable choices of $(\bar{M}, \Lambda)$ render the distribution of F_t tractable enough to be useful for efficient pricing. It is shown below that this class admits several desirable properties for the modeling of joint defaults. More precisely, it contains the full flexibility of the Archimedean class, inherits the singular component of the Marshall-Olkin class, combines the positive dependence coefficients of both classes of copulas, and improves the dynamic aspects of the original Lévy model. An important property of this model is that the resulting dependence structure can be identified, it is the exchangeable Archimax family. The specific form of the survival copula is provided in Theorem 3.1 below, a proof is given in the Appendix.

Theorem 3.1 (The survival copula of the vector of default times)
The survival copula of the vector $(\tau_1, \ldots, \tau_d)'$ *is*

$$\hat{C}(u_1, \ldots, u_d) = \varphi\Big(\frac{1}{\Psi(1)} \sum_{i=1}^{d} \varphi^{-1}(u_{(i)}) \big(\Psi(i) - \Psi(i-1)\big)\Big), \tag{10}$$

where $u_{(1)} \leq \ldots \leq u_{(d)}$ *denotes the ordered list of* $u_1, \ldots, u_d \in [0, 1]$.

Properties of the model

(Sep) A separation of marginals from dependence structure is model inherent, since

$$\begin{aligned}\mathbb{E}[F_t] &= 1 - \mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(- \Lambda_{\bar{M}\,\varphi^{-1}\left(1-p(t)\right)/\Psi(1)}\Big)\Big|\bar{M}\Big]\Big] \\ &= 1 - \mathbb{E}\Big[\exp\Big(- \bar{M}\,\varphi^{-1}\big(1 - p(t)\big)\Big)\Big] = p(t), \quad t \geq 0.\end{aligned}$$

(Cop) The default times have a scale mixture of Marshall-Olkin copulas as survival copula. Statistical properties of this class are investigated, e.g., in [Li 2009]. The specific form is computed in Theorem 3.1.

(Exc) Multiple defaults are possible, since $\{F_t\}_{t\geq 0}$ might have jumps. In the language of copula theory, the survival copula (10) has a singular component on the diagonal. More precisely, the events $\{\tau_1 = \ldots = \tau_k\}$, $k = 2, \ldots, d$, are independent of the realization of $\bar{M}$. Hence, their probabilities solely depend on the Lévy subordinator and can be extracted from a computation in [Mai, Scherer 2009b]:

$$\mathbb{P}(\tau_1 = \ldots = \tau_k) = \frac{\sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \Psi(i)}{\Psi(k)}, \quad k = 2, \ldots, d.$$

($\mathrm{Fs}_{\odot}$) The process $\{F_t\}_{t\geq 0}$ is driven by a mixture of Lévy subordinators. Since $\bar{M}$ is independent of the time t, no time-inhomogeneity is introduced to $\{F_t\}_{t\geq 0}$.

(Tdc) It is not difficult to compute the tail dependence of a pair $(\tau_i, \tau_j)'$ of default times. Using L'Hospital's rule, it is given by[11] (whenever this limit exists)

$$\lambda_l = \lim_{t\downarrow 0} \frac{\mathbb{E}\big[F_t^2\big]}{\mathbb{E}[F_t]} = 2 - \frac{\Psi(2)}{\Psi(1)} \lim_{t\downarrow 0} \frac{\varphi'\big(t\,\Psi(2)/\Psi(1)\big)}{\varphi'(t)}.$$

(Den) Assume that Λ_t admits a density $f_t^{(\Lambda)}$ for $t > 0$. It follows from Fubini's Theorem that the random variable $\Lambda_{\bar{M}\,t}$, $t > 0$, has the density f_t, given by

$$f_t(x) = \mathbb{E}\Big[f^{(\Lambda)}_{\bar{M}\,t}(x)\Big] = \int_0^\infty f^{(\Lambda)}_{y\,t}(x)\,\mathbb{P}(\bar{M} \in dy), \quad x > 0.$$

The latter integral can efficiently be computed when $\bar{M}$ admits a density.

We close this section by giving two specifications of $(\bar{M}, \Lambda)$, which imply viable formulas for all required quantities.

Example 3.2 (An Archimedean model with Armageddon-scenario)
A new parametric family is obtained when an Archimedean model is combined with a Lévy subordinator that increases linearly with drift $\alpha \in [0, 1)$ and might jump to infinity, i.e. its Lévy measure is determined by $\nu(\{\infty\}) = (1-\alpha)$, $\nu((0,\infty)) = 0$. Put differently,

$$\Lambda_t := \alpha\,t + \infty \cdot \mathbb{1}_{\{t>E\}}, \quad t \geq 0,$$

where E is an exponential random variable with mean $1/(1-\alpha)$. Interpreted from an economic point of view, this corresponds to an Archimedean-type dependence structure combined with the positive probability of an Armageddon-scenario killing all remaining components. The resulting survival copula of default times interpolates between the co-monotonicity copula and the chosen Archimedean copula and is given by

$$\hat{C}(u_1,\ldots,u_d) = \varphi\Big((1-\alpha)\,\varphi^{-1}(u_{(1)}) + \alpha\sum_{i=1}^d \varphi^{-1}(u_{(i)})\Big), \quad u_1,\ldots,u_d \in [0,1].$$

The required distributions of $\Lambda_{\bar{M}\,t}$, $t > 0$, are found to be:

$$\mathbb{P}(\Lambda_{\bar{M}\,t} = \infty) = \mathbb{P}(\bar{M}\,t > E) = 1 - \varphi\big(t\,(1-\alpha)\big),$$
$$\mathbb{P}(\Lambda_{\bar{M}\,t} \leq x) = \mathbb{E}\Big[e^{-(1-\alpha)\,\bar{M}\,t}\,\mathbb{1}_{\{\bar{M}\leq \frac{x}{\alpha\,t}\}}\Big], \quad x \in [0,\infty).$$

[11]This result can be validated for the subclasses of Archimedean and Marshall-Olkin survival copulas: the case $\bar{M} \equiv 1$ gives $\lambda_l = 2 - \Psi(2)/\Psi(1)$, which agrees with the result obtained in the model of [Mai, Scherer 2009a]. Similarly, the case $\Lambda_t = t$, $t \geq 0$, leads to $\lambda_l = 2 - 2\lim_{t\downarrow 0}\varphi'(2\,t)/\varphi'(t)$, which agrees with the tail dependence parameter in the model of [Schönbucher 2002]. Higher-dimensional dependence measures can be retrieved from results in [Li 2009].

Example 3.3 (Gamma scale mixture of exchangeable Cuadras-Augé copulas) *A model specification with explicit distribution of F_t is obtained as follows: let Λ be a Poisson process with intensity $\beta > 0$ and $\bar{M}$ be a $\Gamma(1, 1/\theta)$-distributed random variable. In the language of Archimedean copulas, this correponds to C_φ being a Clayton-copula. It follows that*

$$\mathbb{P}(\Lambda_{\bar{M}\,t} = k) = \frac{(t\,\beta)^k}{k!}\,\mathbb{E}\Big[\bar{M}^k\,e^{-\beta\,t\,\bar{M}}\Big] = \frac{(t\,\beta)^k}{\Gamma(1/\theta)\,k!}\int_0^\infty y^k\,e^{-\beta\,t\,y}\,y^{\frac{1}{\theta}-1}e^{-y}\,dy$$
$$= \frac{(t\,\beta)^k}{\Gamma(1/\theta)\,k!}\Big(\frac{1}{1+\beta\,t}\Big)^{k+\frac{1}{\theta}}\,\Gamma\Big(k+\frac{1}{\theta}\Big), \quad k \in \mathbb{N}_0.$$

This choice corresponds to a Gamma scale mixture of exchangeable Cuadras-Augé copulas, see [Cuadras, Augé 1981] for an introduction to the Cuadras-Augé family. Interpreted differently, it corresponds to a generalization of Clayton copulas. The specific form of the copula is obtained from Theorem 3.1 with $\varphi(t) = (1+t)^{1/\theta}$ and $\Psi(x) = \beta\,\big(1-e^{-x}\big)$.

4 A new model based on CGMY-type processes

In this section we propose a CIID-model which combines the intensity-based approach with the approach of [Mai, Scherer 2009a]. Recall that one shortfall of the former class is the fact that $\{F_t\}_{t\geq 0}$ is continuous. Consequently, it does not support joint defaults. On the other side, the model of [Mai, Scherer 2009a], although supporting jumps of $\{F_t\}_{t\geq 0}$, is based on the somewhat unrealistic lack-of-memory properties (inhereted from Λ being a Lévy process). The idea of this generalization is to combine both approaches to create a model that overcomes both shortcomings and produces realistic default pattern over time. Still, it remains tractable enough to allow for efficient pricing routines without Monte-Carlo techniques. When empirical corporate defaults are monitored over time, two stylized facts are revealed (both might equally be realistic for other types of losses): a) There are time periods with few and time periods with many defaults. In between those periods, there is typically a gradual change from one regime to the other. b) Occasionally, there are times with a sudden peak in the number of corporate defaults. The present model is designed to mimic these properties. To formally define the model, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the following (independent) objects:

- A basic affine process $\{\lambda_t\}_{t\geq 0}$ as given by the SDE (6).
- A Lévy subordinator $0 \neq \Lambda = \{\Lambda_t\}_{t\geq 0}$ with Laplace exponent Ψ.
- A list of iid unit exponential random variables $\epsilon_1, \ldots, \epsilon_d$.

With $M_t := \Lambda_{\int_0^t \lambda_s ds/\Psi(1)}$, $t \geq 0$, the individual default times are defined as

$$\tau_k := \inf\big\{t \geq 0 : M_t \geq \epsilon_k\big\}, \quad k = 1, \ldots, d.$$

Due to the definition of τ via three stochastic objects we term this class of models *triply stochastic*. The according CIID-model stems from

$$F_t := 1 - e^{-M_t}, \quad M_t := \Lambda_{\int_0^t \lambda_s \, ds/\Psi(1)}, \quad t \geq 0.$$

The process $\{M_t\}_{t\geq 0}$ is a time-changed Lévy process in the spirit of [Carr et al. 2003]. The Lévy subordinator $\{\Lambda_t\}_{t\geq 0}$ incorporates jumps into $\{F_t\}_{t\geq 0}$, which corresponds to positive probabilities of joint defaults and excess clustering. This accounts for the occurrence of peaks in the number of defaults. The intensity process $\{\lambda_t\}_{t\geq 0}$ incorporates time-inhomogeneity: the larger λ_t, the larger the probability of defaults over the next instance of time. The stochastic nature of $\{\lambda_t\}_{t\geq 0}$ overcomes the lack-of-memory property of the model presented in [Mai, Scherer 2009a]. An important argument for the use of CGMY-type processes is that their Laplace transform is known in closed form. Thus, numerical pricing routines are available using Laplace-inversion techniques, see [Talbot 1979, Abate et al. 1996, Abate, Valkó 2002, Abate, Valkó 2004], which are much more efficient than Monte-Carlo pricing routines. More clearly, one computes

$$\begin{aligned} \mathbb{E}[F_t] &= 1 - \mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(-\Lambda_{\int_0^t \lambda_s \, ds/\Psi(1)}\Big)\Big| \int_0^t \lambda_s \, ds\Big]\Big] \\ &= 1 - \mathbb{E}\Big[e^{-\int_0^t \lambda_s \, ds}\Big] = 1 - e^{\alpha(1,t)+\beta(1,t)\,\lambda_0} = p(t), \quad t \geq 0, \end{aligned} \tag{11}$$

with functions β and α as given in (8) and (9).

This implies that the marginal default probabilities are equal to the ones in the intensity-based approach. Stated differently, the Lévy subordinator only affects the dependence structure. In this regard, $\{\Lambda_t\}_{t\geq 0}$ is an additional source of frailty, which accounts for excess clustering.

Properties of the model

(Sep) The parameters of $\{\lambda_t\}_{t\geq 0}$ enter the formula for $p(t) = \mathbb{E}[F_t]$. However, the parameters of the jump process $\{\Lambda_t\}_{t\geq 0}$ do not affect $p(t)$, see (11). Consequently, the parameters of the intensity can be calibrated to correlation-insensitive market quotes in a first step, and the remaining parameters of the Lévy subordinator provide additional freedom to calibrate the dependence structure in a second step. Hence, even though (Sep) is not fully valid, one can still apply a two-step calibration routine.

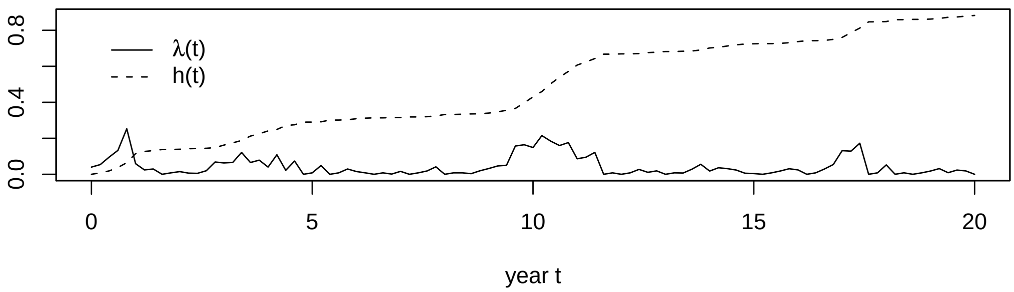
(Cop) Unfortunately, it is not straightforward to compute the multivariate distribution of $(\tau_1, \ldots, \tau_d)'$ in closed form. This complicates the investigation of the underlying dependence structure of the default times.

(Exc) Regarding joint default probabilities, the model inherits all desired properties from the approach of [Mai, Scherer 2009a], since the events $\{\tau_1 = \ldots = \tau_k\}$, for $k =$

$2, \ldots, d$, are independent of the process $\{\lambda_t\}_{t \geq 0}$. In particular, it holds that

$$\mathbb{P}(\tau_1 = \ldots = \tau_k) = \frac{\sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \Psi(i)}{\Psi(k)}, \quad k = 2, \ldots, d.$$

($\mathrm{Fs}_{\oplus}$) The process $\{F_t\}_{t \geq 0}$ is a transformation of a time-changed Lévy subordinator in the spirit of [Carr et al. 2003]. Intuitively, the Lévy subordinator $\{\Lambda_t\}_{t \geq 0}$ accounts for jumps of $\{F_t\}_{t \geq 0}$. Since it is affected by a random time-change, these jumps can occur in a time-inhomogeneous pattern: the larger the intensity λ_t, the more likely a jump of F_t is to occur. This property can be observed in Figure 1, where one realization of the model is illustrated.



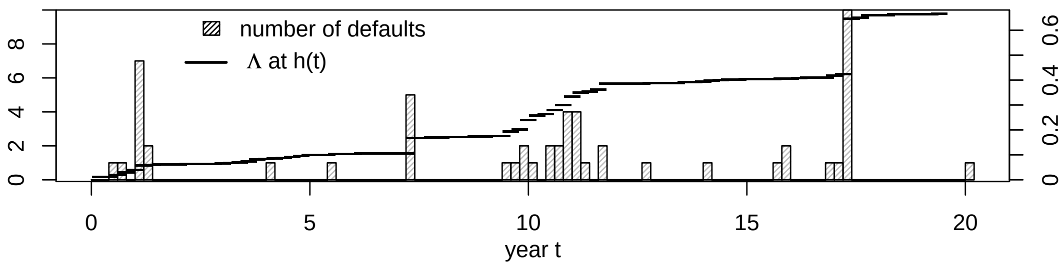


Figure 1 The graph illustrates one realization of the CGMY-based model with portfolio size $d = 125$ over 20 years. The following specifications are made: $\{\lambda_t\}_{t \geq 0}$ is a Cox-Ingersoll-Ross process (a basic affine process without jumps) with $\lambda_0 = 0.04$ and parameters $(\kappa, \theta, \sigma) = (1, 0.04, 0.25)$. The Lévy subordinator is specified by the Laplace exponent $\Psi(x) = x^{0.8}$, $x \geq 0$, i.e. it is a 0.8-stable subordinator. The upper plot illustrates the simulated paths of $\{\lambda_t\}_{t \geq 0}$ and $\{h(t)\}_{t \geq 0}$, where $h(t) := \int_0^t \lambda_s \, ds$. The lower plot illustrates the path of $\{M_t\}_{t \geq 0}$ as well as the observed defaults.

One observes how the realization of the intensity $\{\lambda_t\}_{t \geq 0}$ affects the path of the CGMY-process $\{M_t\}_{t \geq 0}$. In particular, when the intensity is high in the first two years, around year 10 and around year 17, the CGMY-process accumulates many small jumps with default clusters being the consequence. Additionally, the plot visualizes two big jumps of the Lévy subordinator - one around year 7 and one in the middle of the default cluster around year 17. These are interpreted as excess default clustering corresponding to strong and surprising economic shocks.

(Tdc) Starting from (4), a lengthy computation (related to the one in the Appendix for the model without subordinator Λ) involving the specific form of the Laplace transform identifies the lower tail-dependence coefficient of any two default times as $2 - \Psi(2)/\Psi(1)$. This result is intuitive: it agrees with the coefficient in the model of [Mai, Scherer 2009a]. Compared with this model, the deterministic function $t \mapsto h(t)$ is replaced in the present framework by a function of the integrated (random) intensity. However, the intensity-model alone does not generate tail dependence, as shown earlier.

(Den) The density of M_t, $t > 0$, is not known in closed form. However, it can be recovered from its known Laplace transform via numerical Laplace inversion. Using independence of $\{\lambda_t\}_{t\geq 0}$ and $\{\Lambda_t\}_{t\geq 0}$, the Laplace transform is given by

$$\mathbb{E}\Big[\exp\Big(-x\,\Lambda_{\int_0^t \lambda_s\,ds\,/\Psi(1)}\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\frac{\Psi(x)}{\Psi(1)}\int_0^t \lambda_s\,ds\Big)\Big]$$
$$= e^{\alpha\big(\Psi(x)/\Psi(1),t\big)+\beta\big(\Psi(x)/\Psi(1),t\big)\,\lambda_0}, \quad x \geq 0,$$

where α and β are given as in (9) and (8). Since the Lévy subordinator is typically specified in such a way that Ψ has a simple form, the model is as convenient to work with as the classical intensity-based approach.

5 Extensions of the CIID model

This section illustrates classical and new model extensions, formulated in the spirit of the unified stochastic framework of Section 2. This allows the financial engineer to easily adopt extensions from one model class to another. Note, however, that in most cases the convenient large homogeneous portfolio approximation is lost.

5.1 Hierarchical dependence structures

CIID immediately implies exchangeability, an assumption that one might question from an economic point of view. For instance, it is reasonable to assume that companies in the same geographic region (or in the same industry sector) are affected by similar risk factors. Mathematically speaking, to construct a hierarchical model one partitions all firms in J groups - given some economic criterion. Then, all firms are affected by a global factor. In addition to that, specific factors affecting certain subgroups are introduced. Such a model translates into our framework, however, the CIID structure is given up in exchange for a hierarchical model structure. Formally, denote by $d_1, \ldots, d_J$ the number of firms in subgroup j. The default time of company i from subgroup j is denoted τ_{ij} and defined by

$$\tau_{ij} := \inf\big\{t \geq 0 : U_{ij} \leq F_t^{(j)}\big\}, \quad j = 1, \ldots, J,\ i = 1, \ldots, d_j, \tag{12}$$

where $F^{(1)}, \ldots, F^{(J)}$ are group specific frailty distributions and $U_{11}, \ldots, U_{d_J J}$ is a list of iid Uni$[0,1]$-distributed random variables. It is reasonable to assume

$$F^{(j)} = \text{function}_j\big(\{M_t\}_{t\geq 0}, \{M_t^{(j)}\}_{t\geq 0}\big),$$

where $\{M_t\}_{t\geq 0}$ is a global factor and $\{M_t^{(j)}\}_{t\geq 0}$ is specific for group j. Within each group, the resulting dependence structure is again CIID. However, the group specific dependencies might differ from one group to another, since the group specific factors need not be iid. Firms from different groups inherit their dependence structure from the global factor. Note, however, that a large homogeneous portfolio approximation is not available anymore. To work with the model, one must instead rely on Monte-Carlo simulations based on construction (12).

Example 5.1 (Multi-factor Gaussian model)
A well-known example, used in various commercial portfolio default models, is a Gaussian dependence structure implied by a global factor M and, independent thereof, mutually independent group specific factors $M_1, \ldots, M_J$. All factors are standard normal random variables. Then, in the language of Section 2, define with univariate marginal distribution $t \mapsto p(t)$, ρ, $\rho_j \in (0,1)$, and $\rho + \rho_j < 1$

$$F_t^{(j)} := \Phi\Big(\frac{\Phi^{-1}\big(p(t)\big) - \sqrt{\rho}\, M - \sqrt{\rho_j}\, M_j}{\sqrt{1-\rho-\rho_j}}\Big), \quad t \geq 0.$$

The pairwise correlation between two default times of firms from different groups is ρ. The correlation between default times from group j is $\rho + \rho_j$. The random vector of default times has a Gaussian dependence structure with according block-correlation matrix. Extensions to more factors or other factor distributions are immediate.

Example 5.2 (Nested Archimedean copulas)
An elegant example for a hierarchical extension is a nested Archimedean copula. This dependence structure is succesfully applied in the context of CDO pricing models in [Hofert, Scherer (2011)]. Note, however, the very technical notation in this reference. Formulating models based on nested Archimedean copulas in the present language only requires a positive random variable M with Laplace transform φ as global factor and group specific independent Lévy subordinators $\Lambda^{(j)} = \{\Lambda_t^{(j)}\}_{t\geq 0}$, $j = 1, \ldots, J$, with Laplace exponent Ψ_j. Then, the frailty distribution for group j is defined as

$$F_t^{(j)} := 1 - e^{-\Lambda_M^{(j)} \Psi_j^{-1}\left(\varphi^{-1}(1-p(t))\right)}, \quad t \geq 0.$$

Consequently, two default times in group j are coupled by an Archimedean survival copula with generator $\Psi_j(\varphi(x))$, two firms from different groups are coupled by an Archimedean *survival copula with generator $\varphi(x)$, see [Hering et al. (2010)]. One can show that the model matches our intuition in the sense that the dependence within each group is at least as large as the dependence between default times of different groups.*

Example 5.3 (Hierarchical scale mixture of Marshall-Olkin copulas)
A new example for a hierarchical extension is to start with a Lévy subordinator $\Lambda = \{\Lambda_t\}_{t\geq 0}$ as global factor. For each group j, an independent positive random variable M_j with Laplace transform φ_j is considered as additional group-specific factor. Finally, $F^{(j)}$ is defined as

$$F_t^{(j)} := 1 - e^{-\Lambda_{M_j \varphi_j^{-1}\left(1-p(t)\right)/\Psi(1)}}, \quad t \geq 0.$$

5.2 Inhomogeneous marginal distributions

Starting from a CIID model, one possible generalization is to assume conditionally independent (but not necessarily identically distributed) default times; in short, inhomogeneous marginal distributions. Assuming that the marginal default probabilities are model input, i.e. the market frailty is a function of the term structure of default probabilities $t \mapsto p(t)$ and some market factor $M = \{M_t\}_{t\geq 0}$, this is achieved by defining

$$\tau_k := \inf\left\{t \geq 0 : U_k \leq F_t^{(k)} := \text{function}\big(p_k(t), M\big)\right\}, \quad t \geq 0,$$

where the marginal distribution function $\mathbb{E}\big[F_t^{(k)}\big] = p_k(t)$ is now specific to obligor k. In this case, it is still possible to exploit the fact that the resulting default times are conditionally independent. For instance,

$$\mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) = \mathbb{E}\big[\mathbb{E}\big[\prod_{k=1}^{d} \mathbb{1}_{\{\tau_k \leq t_k\}} \big| M\big]\big] = \mathbb{E}\big[\prod_{k=1}^{d} F_{t_k}^{(k)}\big], \quad t_1, \ldots, t_d \geq 0.$$

One way to obtain the loss distribution in this case is to use the classical recursion formula, see, e.g., [Andersen et al. 2003, Dobránszky, Schoutens 2009], adapted to the present framework:

$$\Pi_k^{M,n+1}(t) := \big(1 - F_t^{(n+1)}\big)\Pi_k^{M,n}(t) + F_t^{(n+1)}\Pi_{k-1}^{M,n}, \quad t \geq 0,\, 0 \leq n \leq d-1,$$

where at the end of the iteration, $\Pi_k^{M,d}(t)$ denotes the conditional probability (given M) of having precisely k defaults until time t, where $0 \leq k \leq d$. Note that the iteration must be initialized with $\Pi_0^{M,0} = 1$ and $\Pi_{-1}^{M,d} = 0$. The unconditional probability is obtained by integrating out the market factor. Hence, in order to obtain a viable model, this distribtion must again be tractable. Moreover, this iterative approach becomes slow and prone to rounding errors for large portfolio sizes.

An alternative way to compute the loss distribution is available when the copula behind the default times is known. In this case, it is even possible to compute the probability of having $0 \le k \le d$ distinct defaults up to time t, see, e.g. [Schönbucher 2003, Th. 10.6]. Then, we have to sum over all d choose k subsets to obtain the probability of having k (not further specified) defaults from d names - which of course, is computationally only possible for small portfolios. Note that this considerably simplifies for exchangeable portfolios - the case we just generalized.

5.3 Model fitting across a term structure of maturities

Standardized CDO contracts are traded with maturities 3, 5, 7, and 10 years, respectively, the most liquid ones being maturities of 5 and 10 years. To price contracts with non-standard maturities consistent to market data, one has to match model and market prices across all CDO tranches and across all traded maturities. Since the latter is especially demanding, most investors fix some maturity and fit their model to the tranches for this maturity. However, proceeding like this for each maturity leads to different model specifications - one for each maturity - and it is not clear how to obtain arbitrage-free prices for other maturities. We now show that the present setup is well-suited to allow for a bootstrapping-like routine to fit the model across all maturities, starting with the shortest maturity, and ending with the longest. To describe it, assume a given tenor structure $0 < T_1 < T_2 < \ldots < T_K$ of maturities for which CDO quotes are available. The fundamental idea is to partition the frailty distribution into distinct pieces on the intervals $[0, T_1], (T_1, T_2], \ldots, (T_{K-1}, T_K]$ and to iteratively extend the fit of the CIID-model to the next maturity. Each piece might be interpreted as a *local frailty distribution*. Therefore, the model consists of K (stochastically independent) market frailties. Combining them to an overall stochastic process $\{F_t\}_{t\ge 0}$, such that the resulting model is maturity-consistent, is done in Lemma 5.4 below, the proof is given in the Appendix.

Lemma 5.4 (Bootstrapping CIID structures across different maturities)
Given K stochastically independent market frailties

$$\{F_t^{[0,T_1]}\}_{t\ge 0}, \{F_t^{(T_1,T_2]}\}_{t\ge 0}, \ldots, \{F_t^{(T_{K-1},T_K]}\}_{t\ge 0},$$

we iteratively define the stochastic process $\{F_t\}_{t\ge 0}$ as follows: on $t \in [0, T_1]$, we let $F_t^{(1)} := F_t^{[0,T_1]}$. For $k = 2, \ldots, K$, we then let

$$F_t^{(k)} := \mathbb{1}_{\{t\in[0,T_{k-1}]\}}\, F_t^{(k-1)} + \mathbb{1}_{\{t\in(T_{k-1},T_k]\}}\, F_{T_{k-1}}^{(k-1)} \Big(1 + \frac{1 - F_{T_{k-1}}^{(k-1)}}{F_{T_{k-1}}^{(k-1)}}\, F_{t-T_{k-1}}^{(T_{k-1},T_k]}\Big), \quad t \in [0, T_k].$$

Finally, $F_t := F_t^{(K)}$ for $t \ge 0$ is an admissible market frailty, i.e. a proper distribution function for each $\omega \in \Omega$.

Lemma 5.4 guarantees the validity of an iterative calibration of a CIID-model to CDO quotes referring to different maturities $T_1 < \ldots < T_K$. Recall that the CIID-model implied pricing formulas corresponding to quotes for maturity T_k involve expectation values of the form $\mathbb{E}[f(F_t)]$ for time points $t \leq T_k$. With $\{F_t\}_{t\geq 0}$ being specified such as in Lemma 5.4, it follows that these expectation values only depend on the stochastic factors $\{F_t^{[0,T_1]}\}_{t\geq 0}$, $\{F_t^{(T_1,T_2]}\}_{t\geq 0}, \ldots, \{F_t^{(T_{k-1},T_k]}\}_{t\geq 0}$. By iteration, the parameters of the first $k-1$ factors $\{F_t^{[0,T_1]}\}_{t\geq 0}$, $\{F_t^{(T_1,T_2]}\}_{t\geq 0}, \ldots, \{F_t^{(T_{k-2},T_{k-1}]}\}_{t\geq 0}$ are already determined. Therefore, it is straightforward to calibrate the parameters of the k-th factor $\{F_t^{(T_{k-1},T_k]}\}_{t\geq 0}$ to market quotes of maturity T_k. Depending on the specific forms of the stochastic factors, the expectation values $\mathbb{E}[f(F_t)]$ can either be computed analytically or must be solved via Monte Carlo simulations. As an example, consider three different maturities, say 5, 7, and 10 years, and specify each of the three factors $\{F_t^{[0,5]}\}_{t\geq 0}$, $\{F_t^{(5,7]}\}_{t\geq 0}$, and $\{F_t^{(7,10]}\}_{t\geq 0}$ like in the Archimedean model of Section 2.4. This means that we have three independent, absolutely continuous and positive random variables $M_{[0,5]}$, $M_{(5,7]}$, and $M_{(7,10]}$, with corresponding Laplace transforms $\varphi_{[0,5]}$, $\varphi_{[5,7]}$, and $\varphi_{[5,10]}$. To calibrate the model to CDO quotes corresponding to contracts maturing in 5 years, all involved expectation values are integrals w.r.t. the density of $M_{[0,5]}$. Proceeding with the next maturity of 7 years, all involved expectation values are double integrals w.r.t. the product of (independent) densities of $M_{[0,5]}$ and $M_{(5,7]}$. For the third maturity, we then need to evaluate triple integrals, which is of course much more computationally burdensome. Nevertheless, conceptually the routine is straightforward and the number of different maturities considered in real-live is typically amongst $2, \ldots, 5$, rendering this effort acceptable, in particular because a simultaneous fit of only one market frailty across different maturities is typically not satisfying.

5.4 Incorporation of a stochastic recovery rate

Although common practice, the assumption of constant recoveries is unsatisfactory, since empirical observations suggest a negative association between default intensities and recovery rates, see e.g. [Höcht (2010)]. However, there is a simple approach to incorporate stochastic recovery rates into a CIID framework without giving up its analytical viability. Assuming identical recovery rates for all issuers, one may define the global stochastic recovery rate as a function of the market frailty in such a way that the aforementioned negative dependence is guaranteed. Giving an example, in the Gaussian copula model such an approach is introduced by [Andersen, Sidenius (2004,2005)]. Recalling that the market frailty in the Gaussian CIID-model is given by

$$F_t := \Phi\Big(\frac{\Phi^{-1}\big(p(t)\big) - \sqrt{\rho}\, M}{\sqrt{1-\rho}}\Big), \quad t \geq 0,$$

they propose to define the stochastic recovery rate as

$$R := R_{max}\,\Big(1 - \Phi\Big(\frac{\mu + b\, M}{\sqrt{1+\sigma^2}}\Big)\Big),$$

for parameters $\mu, b \in \mathbb{R}$, $R_{max} \in [0,1]$, and $\sigma > 0$. Generally speaking, whenever the market frailty process $\{F_t\}_{t\geq 0}$ is a function of a random variable (or vector) M, i.e. $F_t := f(M,t)$, defining $R := g(M)$ as a function of M implies that uniformly in $t \geq 0$ we have almost surely

$$(1-R)\,L_t \;\rightarrow\; \big(1-g(M)\big)\,f(M,t), \quad \text{as } d \rightarrow \infty.$$

Hence, the large homogeneous portfolio approximation is still possible and the limiting variable is again a function of M, and, thus, typically convenient to handle. To achieve the desired negative dependence between recovery rates and default intensities, the appropriate choice of g is essential. One must guarantee that $f(M,t)$ is increasing in M while $g(M)$ is decreasing in M, or vice versa. This reflects the fact that increasing default likelihood corresponds to decreasing recovery rates, and vice versa. This technique is in principle applicable to the models in Sections 2.3 and 2.4. For the more sophisticated model specifications based on a stochastic process $\{M_t\}_{t\geq 0}$ rather than a random variable M, it is a more delicate issue to incorporate stochastic recovery rates in a reasonable way. This is subject of further research.

6 Conclusion

A unified approach for CIID portfolio default models was presented. Desirable stochastic properties of these models were introduced in an axiomatic manner and discussed from an economic perspective, mostly with a view on credit risk. State-of-the-art models that fit into the present context were discussed and compared with respect to these properties. Two new models, extending several well-known models, were introduced. The first one was shown to unify the approaches of [Schönbucher 2002] and [Mai, Scherer 2009a]. The resulting implied copula is of Archimax type. The second ansatz combines a classical intensity approach with a Lévy based approach to allow for excess clustering and time-inhomogeneity. In both cases one could derive the Laplace transform of the required underlying frailty distribution in closed form. Finally, several model generalizations are discussed.

Appendix

Lévy subordinators

A Lévy subordinator $\Lambda = \{\Lambda_t\}_{t\geq 0}$ is a non-decreasing stochastic process. It starts at zero, is stochastically continuous, and has stationary and independent increments. Standard textbooks on this theory comprise [Bertoin 1996, Bertoin 1999, Sato 1999, Schoutens 2003, Applebaum 2004]. A Lévy subordinator is uniquely characterized by its Laplace transforms, which admit the form

$$\mathbb{E}\big[e^{-x\,\Lambda_t}\big] = e^{-t\,\Psi(x)}, \quad \forall\, x \geq 0,\, t \geq 0,$$

for a function $\Psi : [0, \infty) \rightarrow [0, \infty)$ which has a completely monotone derivative and satisfies $\Psi(0) = 0$, see [Feller 1966, p. 450]. The function Ψ is called *Laplace exponent* of Λ and is strictly increasing unless $\Lambda_t \equiv 0$.

Copulas

A (d-dimensional) *copula* C is the restriction on $[0,1]^d$ of the distribution function of a random vector $(U_1, \ldots, U_d)^{'}$, where each U_k is Uni$[0,1]$-distributed. If $(\tau_1, \ldots, \tau_d)^{'}$ is an arbitrary random vector with continuous marginal distribution functions $t \mapsto F_k(t) := \mathbb{P}(\tau_k \leq t)$, $k = 1, \ldots, d$, then the distribution function of the random vector $\big(F_1(\tau_1), \ldots, F_d(\tau_d)\big)^{'}$ is a copula, called the *copula* of $(\tau_1, \ldots, \tau_d)^{'}$. Similarly, the distribution function of the random vector $\big(1 - F_1(\tau_1), \ldots, 1 - F_d(\tau_d)\big)^{'}$ is a copula, called the *survival copula* of $(\tau_1, \ldots, \tau_d)^{'}$. The investigation of the distribution of $(\tau_1, \ldots, \tau_d)^{'}$ can be split into two substudies (marginals and dependence structure) by virtue of Sklar's Theorem, see [Sklar 1959]: if C (resp. $\hat{C}$) denotes the copula (resp. survival copula) of $(\tau_1, \ldots, \tau_d)^{'}$, then it holds true that

$$\mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) = C\big(F_1(t_1), \ldots, F_d(t_d)\big), \quad t_1, \ldots, t_d \in \mathbb{R},$$
$$\mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \hat{C}\big(1 - F_1(t_1), \ldots, 1 - F_d(t_d)\big), \quad t_1, \ldots, t_d \in \mathbb{R}.$$

In the present context such a separation of marginals and dependence structure is useful, since the marginal distributions of default times are often given a priori. In particular, the model for the marginals and the copula are typically different, which necessitates an intrinsic study of the underlying copula. Depending on the specific distribution, sometimes it is easier to explore the copula, sometimes it is easier to explore the survival copula. Considering specific families, an *elliptical copula* corresponds to the respective elliptical distribution by standardizing its marginals. Elliptical distributions, in turn, are defined as linear transformations of spherical distributions, see [Fang et al. 1989, p. 31]. Prominent members of this class include the Gaussian copula and the Student t-copula. A copula C_φ is called *Archimedean* with generator φ if it has the functional form

$$C_\varphi(u_1, \ldots, u_d) = \varphi\big(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_d)\big), \quad u_1, \ldots, u_d \in [0,1].$$

It is shown in [Marshall, Olkin 1988] that if $\varphi(x) := \mathbb{E}[\exp(-x\, M)]$, $x \geq 0$, is the Laplace transform of a positive random variable M, the random vector $(\epsilon_1/M, \ldots, \epsilon_d/M)^{'}$ has survival copula C_φ, where $\epsilon_1, \ldots, \epsilon_d$ are iid unit exponential, independent of M. This means that Archimedean copulas occur as mixture distributions of iid exponential random variables whose parameter M is randomly chosen. The so-called *Marshall-Olkin distribution* is introduced in [Marshall, Olkin 1967] as the unique multivariate extension of the exponential distribution with lack-of-memory property. Additional references include, e.g., [Barlow, Proschan 1975, Galambos, Kotz 1978, Mai, Scherer 2011, Mai, Scherer 2009b]. The motivation behind this distribution is an exogenous shock model: in dimension $d \geq 2$, consider $2^d - 1$ independent random variables ϵ_I, $\emptyset \neq I \subset \{1, \ldots, d\}$, where

$\epsilon_I \sim Exp(\lambda_{|I|})$ for parameters $\lambda_1, \ldots, \lambda_d > 0$. The random vector $(\tau_1, \ldots, \tau_d)'$, defined by

$$\tau_k := \min_{I \subseteq \{1,\ldots,d\}} \big\{\epsilon_I \,\big|\, k \in I\big\}, \quad k = 1, \ldots, d,$$

is said to have the *exchangeable Marshall-Olkin distribution*. Its survival copula has the form

$$\hat{C}(u_1, \ldots, u_d) = \prod_{k=1}^{d} u_{(k)}^{\frac{\sum_{i=0}^{d-k} \binom{d-k}{i} \lambda_{i+1}}{\sum_{i=0}^{d-1} \binom{d-1}{i} \lambda_{i+1}}}, \quad u_1, \ldots, u_d \in [0,1], \tag{13}$$

where $u_{(1)} \leq \ldots \leq u_{(d)}$ denotes the ordered list of $u_1, \ldots, u_d$, see [Mai, Scherer 2011]. An important fact for our purpose is the following, see [Mai, Scherer 2011]: if Λ is a Lévy subordinator with Laplace exponent Ψ and $\epsilon_1, \ldots, \epsilon_d$ are iid unit exponential and independent of Λ, then the random vector $(\tau_1, \ldots, \tau_d)'$ defined by

$$\tau_k := \inf\{t > 0 \,:\, \Lambda_t \geq \epsilon_k\}, \quad k = 1, \ldots, d,$$

has the exchangeable Marshall-Olkin distribution with parameters given by

$$\lambda_k := \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \big(\Psi(d-k+i+1) - \Psi(d-k+i)\big), \quad k = 1, \ldots, d.$$

In particular, its survival copula is given as in (13) and simplifies to

$$\hat{C}(u_1, \ldots, u_d) = \prod_{k=1}^{d} u_{(k)}^{\big(\Psi(k)-\Psi(k-1)\big)/\Psi(1)}, \quad u_1, \ldots, u_d \in [0,1].$$

Proof of Lemma 2.2

The first statement follows immediately from the Theorem of Glivenko-Cantelli, see [Loève 1977, p. 20]: conditioned on $\{F_t\}_{t\geq 0}$, $\{L_t\}_{t\geq 0}$ is precisely the empirical distribution function of the law $\{F_t\}_{t\geq 0}$ based on d samples. Hence,

$$\mathbb{P}\Big(\lim_{d\to\infty} \sup_{t\geq 0} |F_t - L_t| = 0\Big) = \mathbb{E}\Big[\mathbb{P}\Big(\lim_{d\to\infty} \sup_{t\geq 0} |F_t - L_t| = 0 \,\Big|\, \{F_t\}_{t\geq 0}\Big)\Big] = \mathbb{E}[1] = 1.$$

For the second statement, immediate computations show that

$$\mathbb{E}\big[L_t^2\big] = \frac{1}{d}\,\mathbb{E}[F_t] + \frac{d-1}{d}\,\mathbb{E}[F_t^2], \quad \mathbb{E}[L_t\, F_t] = \mathbb{E}[F_t^2],$$

which implies that

$$\int_{[0,T]} \mathbb{E}\big[(L_t - F_t)^2\big]\, dt = \frac{1}{d} \int_{[0,T]} \big(\mathbb{E}[F_t] - \mathbb{E}[F_t^2]\big)\, dt \quad \stackrel{d\to\infty}{\longrightarrow} \quad 0.$$

The claim is established. □

Proof of zero tail dependence in the model of Section 2.5

The first step is to compute for β and α, given in (8) and (9), that

$$\alpha'(x,0) := \lim_{t\downarrow 0}\frac{d}{dt}\alpha(x,t) \overset{(*)}{=} 0, \quad \beta'(x,0) := \lim_{t\downarrow 0}\frac{d}{dt}\beta(x,t) \overset{(**)}{=} -x, \quad x>0.$$

Both $(*)$ and $(**)$ are tedious computations that become simpler with the identities

$$c(x)+d(x) = -\frac{\sqrt{\kappa^2+2\sigma^2 x}}{x}, \quad c(x)-d(x) = -\frac{\kappa}{x},$$
$$c(x)-d(x) = \frac{\sigma^2}{2x}, \quad \frac{b(x)}{c(x)+d(x)} = x.$$

Then, using formula (7), one computes with L'Hospital's rule that[12]

$$\lambda_l = \lim_{t\downarrow 0}\frac{\mathbb{E}\big[F_t^2\big]}{\mathbb{E}[F_t]} = \lim_{t\downarrow 0}\left\{1+\frac{e^{\alpha(2,t)+\beta(2,t)\,\lambda_0}-e^{\alpha(1,t)+\beta(1,t)\,\lambda_0}}{1-e^{\alpha(1,t)+\beta(1,t)\,\lambda_0}}\right\}$$
$$= 2-\frac{\alpha'(2,0)+\beta'(2,0)\,\lambda_0}{\alpha'(1,0)+\beta'(1,0)\,\lambda_0} = 0.$$

Proof of Theorem 3.1

At first, observe that for arbitrary $t_1,\ldots,t_d \in [0,\infty)$ with ordered list $t_{(1)} \le \ldots \le t_{(d)}$ and $t_{(0)} := 0$ it holds that

$$\sum_{i=1}^{d}(d+1-i)\left(\Lambda_{t_{(i)}}-\Lambda_{t_{(i-1)}}\right) = \sum_{i=1}^{d}(d+1-i)\,\Lambda_{t_{(i)}} - \sum_{i=0}^{d-1}(d-i)\,\Lambda_{t_{(i)}} = \sum_{i=1}^{d}\Lambda_{t_i}.$$

Since Λ is a Lévy process, the vector of increments $\left(\Lambda_{t_{(d)}}-\Lambda_{t_{(d-1)}},\ldots,\Lambda_{t_{(1)}}-\Lambda_{t_{(0)}}\right)$ has independent components and $\Lambda_{t_{(i)}}-\Lambda_{t_{(i-1)}}$ is equal in distribution to $\Lambda_{t_{(i)}-t_{(i-1)}}$. Consequently

$$\mathbb{E}\left[e^{-\sum_{i=1}^{d}\Lambda_{t_i}}\right] = \prod_{i=1}^{d}\mathbb{E}\left[e^{-(d+1-i)\,\Lambda_{(t_{(i)}-t_{(i-1)})}}\right] = \prod_{i=1}^{d}e^{-(t_{(i)}-t_{(i-1)})\,\Psi(d+1-i)}.$$

[12] The above argument makes use of the explicit form of the Laplace transform of an integrated basic affine intensity. If the intensity $\{\lambda_t\}_{t\ge 0}$ is specified differently, one might end up with positive tail dependence. Giving one example, assume that $\lambda_t := \bar{M}$ for a positive random variable $\bar{M}$ with Laplace transform φ, i.e. $M_t = \bar{M}\,t$, $t\ge 0$. The resulting dependence structure is of Archimedean kind and there are choices for $\bar{M}$ that imply positive tail dependence. A related observation is that for $\lambda_t := \bar{M}\,\frac{\partial}{\partial t}\big(\varphi^{-1}(1-p(t))\big)$, the model of [Schönbucher 2002] is a special case of the intensity approach.

Secondly, compute the joint survival function (using the above identity)

$$
\begin{aligned}
G(t_1,\dots,t_d) &:= \mathbb{P}\big(\tau_1 > t_1,\dots,\tau_d > t_d\big) \\
&= \mathbb{P}\big(\epsilon_1 > \Lambda_{\bar{M}\,\varphi^{-1}(1-p(t_1))/\Psi(1)},\dots,\epsilon_d > \Lambda_{\bar{M}\,\varphi^{-1}(1-p(t_d))/\Psi(1)}\big) \\
&= \mathbb{E}\Bigg[\exp\Big(-\sum_{i=1}^{d}\Lambda_{\bar{M}\,\varphi^{-1}(1-p(t_i))/\Psi(1)}\Big)\Bigg] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\exp\Big(-\sum_{i=1}^{d}\Lambda_{\bar{M}\,\varphi^{-1}(1-p(t_i))/\Psi(1)}\Big)\Big|\bar{M}\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[\exp\Big(-\frac{\bar{M}}{\Psi(1)}\sum_{i=1}^{d}\Psi(d+1-i)\big(\varphi^{-1}(1-p(t_{(i)}))-\varphi^{-1}(1-p(t_{(i-1)}))\big)\Big)\Bigg] \\
&= \varphi\Big(\frac{1}{\Psi(1)}\sum_{i=1}^{d}\Psi(d+1-i)\big(\varphi^{-1}(1-p(t_{(i)}))-\varphi^{-1}(1-p(t_{(i-1)}))\big)\Big) \\
&= \varphi\Big(\frac{1}{\Psi(1)}\sum_{i=1}^{d}\varphi^{-1}(1-p(t_{(d+1-i)}))\big(\Psi(i)-\Psi(i-1)\big)\Big).
\end{aligned}
$$

The last step involves expanding the sum of differences to two sums and shifting the summation index in the second sum by one. The resulting sums can then be recombined using $\Psi(0) = 0$.

Thirdly, for the margins one obtains using similar arguments

$$\mathbb{P}\big(\tau_i > t\big) = \mathbb{P}\big(\epsilon_i > \Lambda_{\bar{M}\,\varphi^{-1}(1-p(t))/\Psi(1)}\big) = 1 - p(t), \quad i = 1,\dots,d, \quad t \geq 0.$$

Thus, τ_i is distributed according to $p(t)$. Finally, using the survival analogue of Sklar's Theorem, see [McNeil et al. 2005, p. 195], there exists a unique copula $\hat{C}$, called the survival copula of $(\tau_1,\dots,\tau_d)$, which satisfies

$$G(t_1,\dots,t_d) = \hat{C}\big(1-p(t_1),\dots,1-p(t_d)\big).$$

Testing the copula claimed in (10), one finds

$$\hat{C}\big(1-p(t_1),\dots,1-p(t_d)\big) = \varphi\Big(\frac{1}{\Psi(1)}\sum_{i=1}^{d}\varphi^{-1}(1-p(t_{(d+1-i)}))\big(\Psi(i)-\Psi(i-1)\big)\Big).$$

Thus, the claim is established by the uniqueness of the survival copula. □

Proof of Lemma 5.4

The claimed composition of different distribution functions to a new one is based on an elementary decomposition of a distribution function. Considering only two intervals

$[0, T_1]$ and $(T_1, T_2]$, it is easy to verify for $t \in [0, T_2]$ that

$$\mathbb{P}(\tau \leq t) = \mathbb{1}_{\{t \in [0,T_1]\}} \, \mathbb{P}(\tau \leq t) + \mathbb{1}_{\{t \in (T_1,T_2]\}} \, \mathbb{P}(\tau \leq T_1) \left(1 + \frac{\mathbb{P}(\tau > T_1)}{\mathbb{P}(\tau \leq T_1)} \, \mathbb{P}(\tau \leq t \,|\, \tau > T_1)\right).$$

The crucial observation from this elementary computation is that having determined the distribution $p_1(t) := \mathbb{P}(\tau \leq t)$ on $[0, T_1]$ already, to determine the distribution on $[0, T_2]$ it suffices to determine $p_2(u) := \mathbb{P}(\tau \leq u + T_1 \,|\, \tau > T_1)$ for $u \in (0, T_2 - T_1]$. However, the function p_2 is a proper distribution function on $[0, \infty)$. Hence, starting with two given distribution functions p_1, p_2, the claimed composition of those yields a proper distribution function with the interpretation that p_2 is the conditional distribution in case of survival until time T_1. The general case $K > 2$ is now easily obtained by iterating the above argument.

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