On Two-Stage Fixed-Width Confidence Interval Procedures for the Mean of a Normal Distribution

Mukhopadhyay, Nitis
University of Connecticut, Department of Statistics
U-4120, CLAS Bldg., 215 Glenbrook Road
Storrs, CT 06269-4120, USA
E-mail: *nitis.mukhopadhyay@uconn.edu*

Chattopadhyay, Bhargab
University of Connecticut, Department of Statistics
U-4120, CLAS Bldg., 215 Glenbrook Road
Storrs, CT 06269-4120, USA
E-mail: *bhargab.chattopadhyay@uconn.edu*

1. Introduction

Designs of original two-stage and multi-stage sampling methodologies and their practical implementations in large-scale sample surveys date back to Mahalanobis's (1940) pioneering research. The broad area of multi-stage and sequential estimation problems may be reviewed from Sen (1981), Woodroofe (1982), Siegmund (1985), Mukhopadhyay and Solanky (1994), Ghosh and Sen (1990), Ghosh et al. (1997), Mukhopadhyay et al. (2004), and Mukhopadhyay and de Silva (2009) among other sources.

Let us begin by assuming availability of a sequence $X_1, X_2, ...$ of independent observations following a normal distribution with unknown mean μ and unknown variance $\sigma^2, -\infty < \mu < \infty, 0 < \sigma < \infty$. Having recorded $X_1, ..., X_n$, let us denote the customary estimators:

$$\begin{array}{ll} \text{Sample Mean:} & \overline{X}_n = n^{-1}\Sigma_{i=1}^n X_i \\ \text{Sample Variance:} & S_n^2 = (n-1)^{-1}\Sigma_{i=1}^n (X_i - \overline{X}_n)^2, n \geq 2. \end{array} \quad (1.1)$$

Had σ^2 being known, the optimal fixed sample size needed to have a preassigned fixed-width $(= 2d)$ confidence interval centered at the sample mean would be $C \equiv a^2\sigma^2/d^2$ where $d(> 0)$ is half-length of the confidence interval and a is the upper $50\alpha\%$ point of $N(0, 1)$. But σ^2 is unknown, and so one would customarily estimate C using the sample variance.

Stein's (1945,1949) fundamental two-stage procedure for constructing a fixed-width confidence interval for μ involved S_m^2 obtained from pilot observations $X_1, ..., X_m$ with the pilot size $m(\geq 2)$ and a replaced with $t_{m-1,\alpha/2}$. One may also refer to Cox (1952). Mukhopadhyay (1982) opened the possibility of incorporating less traditional estimators of σ^2.

In this note, we focus on estimating σ^2 by a statistic U_m^2 defined via *mean absolute deviation* (MAD), *range*, and *Gini's mean difference*. Obviously, then, $t_{m-1,\alpha/2}$ must be replaced by the upper $50\alpha\%$ point corresponding to a pivotal distribution of the sample mean standardized by U_m. For brevity, let us state specifically some of such estimators :

$$\begin{array}{lll} \text{Sample Var:} & S_n^2 & \widehat{\sigma}^2 \equiv U_n^{(1)2} = S_n^2 \\ \text{MAD:} & M_n = n^{-1}\Sigma_{i=1}^n \left|X_i - \overline{X}_n\right| & \widehat{\sigma}^2 \equiv U_n^{(2)2} = M_n^2/c_n^{(2)2} \\ \text{Gini:} & G_n = \binom{n}{2}^{-1} \underset{1\le i<j\le n}{\Sigma\Sigma} \left|X_i - X_j\right| & \widehat{\sigma}^2 \equiv U_n^{(3)2} = G_n^2/c_n^{(3)2} \\ \text{Range:} & R_n = X_{n:n} - X_{n:1} & \widehat{\sigma}^2 \equiv U_n^{(4)2} = R_n^2/c_n^{(4)2} \end{array} \quad (1.2)$$

where $X_{n:i}$ is the i^{th} order statistic from $X_1, ..., X_n$ and

$$\begin{aligned} c_n^{(2)} &= \left\{\left(\tfrac{\pi}{2} + \sin^{-1}\left(\tfrac{1}{n-1}\right) - n + \sqrt{n(n-2)}\right)\tfrac{2(n-1)}{n^2\pi} + \tfrac{2(n-1)}{\pi n}\right\}^{1/2}, \\ c_n^{(3)} &= \left\{\tfrac{4}{\pi n(n-1)}\left(\tfrac{\pi}{3}(n+1) + 2(n-2)\sqrt{3} + n^2 - 5n + 6\right)\right\}^{1/2}, \\ c_n^{(4)} &= \left\{n(n-1)\int_0^\infty w^2\left(\int_{-\infty}^\infty [\Phi(x+w) - \Phi(x)]^{m-2}\phi(x)\phi(x+w)dx\right)dw\right\}^{1/2}, \end{aligned} \quad (1.3)$$

where $\phi(x) = (2\pi)^{-1/2}\exp\left(-x^2/2\right), \Phi(x) = \int_{-\infty}^x \phi(y)dy, -\infty < x < \infty$. One may note that the expressions $c_n^{(2)}, c_n^{(3)}, c_n^{(4)}$ given in (1.3) can be found in Herrey (1965), Nair (1936), and Owen (1962, p. 140) respectively among other sources.

Observe that $U_n^{(i)2}$, $i = 1, 2, 3, 4$, are unbiased estimators of σ^2. Even though $U_n^{(1)2}$ is the best unbiased estimator of σ^2, many authors have alternatively incorporated $U_n^{(i)2}$, $i = 2, 3, 4$, instead of $U_n^{(1)2}$ and explored their roles in the context of both estimation and tests of hypotheses. Confidence intervals based on M_n are useful in experimental physics (Herrey, 1965). As a measure of deviation, R_n is widely used in quality control and for on-line positioning user service utility reports where the positioning (geodetic latitude, logitude and elevation/height) data are assumed normally distributed. Lord (1947) discussed the role of the statistic $\sqrt{n}(\overline{X}_n - \mu)/R_n$ as a possible competitor to Student's t test.

Yet another unbiased estimator of the population standard deviation that is a suitable multiple of G_n was originally developed by Gini (1914,1921). Nair (1936) constitutes one of the early contributions which discussed the role of G_n in estimation theory for a normal distribution as well as some other selected non-normal distributions. Both Downton (1966) and D'Agostino (1970) worked with an ordered version of G_n and came up with estimates of population standard deviation in a normal distribution. Barnett et al. (1967) discussed the role of $\sqrt{n}(\overline{X}_n - \mu)/G_n$ as a possible competitor to Student's t test and argued that these tests are nearly equally powerful. Further explorations are immensely valuable in order to guide practical users especially when one could reasonably expect some outlying observations even though the observations follow a normal distribution.

Thus, we explore the role of Mukhopadhyay's (1982) two-stage confidence interval procedure when the requisite sample size is determined through $U_n^{(i)2}$, $i = 2, 3, 4$, along with associated exact and first-order properties. Next, we revisit Mukhopadhyay and Duggan's (1997) updated two-stage methodology that was proposed when a known positive lower bound σ_L^2 was available for σ^2 in the present light with hints of associated exact, first-order and second-order properties. We supplement this ongoing methodological research with data analyses. Some details are indicated, though briefly, in the sections that follow.

2. Formulation

Following the generalization of the Stein (1945,1949) procedure by Mukhopadhyay (1982), a fixed-width confidence interval for μ can be determined by exploiting the pivotal distribution of the random variable $W_m^{(i)} = \sqrt{n}(\overline{X}_n - \mu)/U_m^{(i)}$, $i = 1, 2, 3, 4$, for any $n \geq m$. Note that the distribution of $W_m^{(i)}$, does not involve either n or σ^2. So, let us determine $b_{m;\alpha}^{(i)2}$ in such way that

$$P\left\{\left|W_m^{(i)}\right| \leq b_{m,\alpha}^{(i)}\right\} = 1 - \alpha, i = 1, 2, 3, 4, \tag{2.1}$$

where $m(\geq 2)$ happens to be the fixed pilot sample size.

Now, based on the pilot data $X_1, ..., X_m, m \geq 2$, one would determine the final sample size as

$$N^{(i)} \equiv N^{(i)}(d) = \max\left\{m, \left\langle b_{m,\alpha}^{(i)2} U_m^{(i)2}/d^2 \right\rangle + 1\right\}, i = 1, 2, 3, 4, \tag{2.2}$$

where $< w >$ denotes the largest integer $< w$ with $w > 0$.

If $N^{(i)} = m$, no more observations are recorded beyond the pilot set, but if $N^{(i)} > m$, then we would record $N^{(i)} - m$ additional observations in the second stage. Finally, based on the combined data $X_1, ..., X_{N^{(i)}}$ from both stages, one would propose the fixed-width confidence interval

$$I_{N^{(i)}}^{(i)} = \left[\overline{X}_{N^{(i)}} \pm d\right] \tag{2.3}$$

for μ, $i = 1, 2, 3, 4$.

We must emphasize that Stein's (1945,1949) two-stage procedure developed the final sample size $N^{(1)}$ and the associated fixed-width confidence interval $I_{N^{(1)}}^{(1)}$ for μ. Obviously, in this case, one has $W_m^{(1)} \sim t_{m-1}$ and hence replace $b_{m,\alpha}^{(1)}$ with $t_{m-1,\alpha/2}$.

3. Properties of Competing Confidence Intervals

Now, for the two-stage procedure (2.2)-(2.3), we summarize the following set of both exact and first-order asymptotic results.

Theorem 3.1. *For the two-stage fixed-width confidence interval estimation strategy* $\left(N^{(i)}, I_{N^{(i)}}^{(i)}\right)$ *from (2.2)-(2.3), for all fixed* $(\mu, \sigma) \in \Re \times \Re^+, \alpha \in (0, 1)$, *we have:*

(i) $P_{\mu,\sigma}\left\{\mu \in I_{N^{(i)}}^{(i)}\right\} \geq 1 - \alpha$, *for fixed all fixed* d [*Exact Consistency*];

(ii) $E_{\mu,\sigma}\left[N^{(i)}/C\right] \rightarrow \left(b_{m,\alpha}^{(i)}/a\right)^2$ *as* $d \rightarrow 0$;

(iii) $P_{\mu,\sigma}\left\{\mu \in I_{N^{(i)}}^{(i)}\right\} \rightarrow 1 - \alpha$ *as* $d \rightarrow 0$ [*Asymptotic Consistency*];

with $C = a^2\sigma^2/d^2$, $i = 1, 2, 3, 4$.

Proof: From (2.2), we may write down the following basic inequality:

$$b_{m,\alpha}^{(i)2} U_m^{(i)2} d^{-2} \leq N^{(i)} \leq m + b_{m,\alpha}^{(i)2} U_m^{(i)2} d^{-2}, \text{ w.p.1, } i = 1, 2, 3, 4. \tag{3.1}$$

Next, following Mukhopadhyay (1982), we may write down:

$$\begin{aligned} &P_{\mu,\sigma}\left\{\mu \in I^{(i)}_{N^{(i)}}\right\} \\ &\quad = P_{\mu,\sigma}\left\{\left|\overline{X}_{N^{(i)}} - \mu\right| \le d\right\} \\ &\quad = P_{\mu,\sigma}\left\{\sqrt{N^{(i)}}\left|\overline{X}_{N^{(i)}} - \mu\right| \le d\sqrt{N^{(i)}}\right\} \\ &\quad \ge P_{\mu,\sigma}\left\{\sqrt{N^{(i)}}\left|\overline{X}_{N^{(i)}} - \mu\right| / U^{(i)}_m \le b^{(i)}_{m,\alpha}\right\}, \text{ using lower bound from (3.1).} \end{aligned} \tag{3.2}$$

Now, since $U^{(i)2}_m$ is location invariant and hence $\overline{X}_n, U^{(i)2}_m$ are independently distributed for all fixed $n \ge m$ (Basu, 1955). We may rewrite (3.2) as follows:

$$\begin{aligned} &P_{\mu,\sigma}\left\{\mu \in I^{(i)}_{N^{(i)}}\right\} \\ &\quad \ge \Sigma^{\infty}_{n=m} P_{\mu,\sigma}\left\{\sqrt{n}\left|\overline{X}_n - \mu\right| / U^{(i)}_m \le b^{(i)}_{m,\alpha} \cap N^{(i)} = n\right\} \\ &\quad = \Sigma^{\infty}_{n=m} P_{\mu,\sigma}\left\{\sqrt{n}\left|\overline{X}_n - \mu\right| / U^{(i)}_m \le b^{(i)}_{m,\alpha}\right\} P_{\mu,\sigma}\left\{N^{(i)} = n\right\} \\ &\quad = (1-\alpha) P_{\sigma}\left\{N^{(i)} < \infty\right\} \\ &\quad = (1-\alpha), \end{aligned} \tag{3.3}$$

which proves part (i).

Proofs of Parts (ii) and (iii) are routine. ■

Obviously, all four procedures from (2.2)-(2.3) have the same *exact consistency* or *consistency* property (part (i)) and first-order *asymptotic consistency* property (part (iii)) in the sense of Chow and Robbins (1965), Ghosh and Mukhopadhyay (1981), and Mukhopadhyay (1982). The limiting ratio, namely $\lim_{d \to 0} E_{\mu,\sigma}\left[N^{(i)}/C\right]$, would vary from one procedure to another which is reflected in the expression $\left(b^{(i)}_{m,\alpha}/a\right)^2$ given in part (ii). This limiting ratio would exceed one, that is, all four procedures will be associated with appropriate limiting oversampling rates which are illustrated in Table 3.1.

3.1. Moderate Sample Size Performances Via Simulations

We fixed $m = 10, 15, 20$ and estimated the $100(1 - \frac{1}{2}\alpha)$ percentile point $b^{(i)}_{m,\alpha}$ for the distribution of $W^{(i)}_m, i = 2, 3, 4$ by means of simulation. In a specific situation with the sample size m, we obtained m observations from $N(0, 1)$ under each replication and recorded the associated observed value $w^{(i)}_{mr}$ of $W^{(i)}_m$ during the r^{th} replication, $r = 1, 2, ..., 100, 000$. From such $100, 000$ observed $w^{(i)}_{mr}$ values we obtained the required percentiles $b^{(i)}_{m,\alpha}$ for each $i = 2, 3, 4$. These estimated $b^{(i)}_{m,\alpha}$'s are summarized in Table 3.1. Observe that $b^{(1)}_{m,\alpha}$ is normally read from a t-table.

We fixed $\alpha(= 0.10, 0.05, 0.01)$ and d so that C varied within a large range values. Then, under each configuration, we implemented the two-stage procedures from (2.2)-(2.3) and estimated the average sample size $(\overline{n})$, the maximum sample size $(\max(n))$, the standard error $(s(\overline{n}))$ of $\overline{n}$, and the coverage probability $(\overline{p})$ based on $100, 000$ replications via computer simulations by drawing random samples from a $N(5, 4)$ population. As a representative of our findings, we provide Table 3.1 that summarizes the analyses under one configuration only, namely when $\alpha = 0.10$ and $d = 0.5$ so that $C = 43.29$.

Table 3.1. Estimated Average Final Sample Size, Maximum Sample Size, and Coverage Probability Under Two-Stage Procedure (2.2)-(2.3): $\alpha = 0.10, a = 1.645, d = 0.5, C = 43.29$

m	(2.2) index i	$b_{m,\alpha}^{(i)}$	$\overline{n}$ **s**$(\overline{n})$	$\overline{n}/C$	$b_{m,\alpha}^{(i)2}a^{-2}$	max(n)	$\overline{p}$ **s**$(\overline{p}) \times$ **10**^{4}
10	1	1.833113	54.25	1.2532	1.2418	224	0.90114
			0.07991				**9.4386**
	2	1.856686	55.60	1.2844	1.2739	245	0.90243
			0.08646				**9.3835**
	3	1.840581	54.68	1.2631	1.2519	233	0.90193
			0.08149				**9.4049**
	4	1.867060	56.28	1.3001	1.2882	300	0.90433
			0.09108				**9.3015**
15	1	1.761310	50.21	1.1599	1.1464	180	0.90234
			0.05924				**9.3874**
	2	1.773683	50.88	1.1753	1.1626	192	0.90324
			0.06350				**9.3487**
	3	1.761506	50.22	1.1601	1.1467	190	0.90260
			0.05988				**9.3762**
	4	1.797205	52.31	1.2084	1.1936	229	0.90230
			0.07165				**9.3891**
20	1	1.729133	48.37	1.1173	1.1049	143	0.90280
			0.04897				**9.3676**
	2	1.739828	48.97	1.1312	1.1186	159	0.9039
			0.05265				**9.3201**
	3	1.730803	48.46	1.1194	1.1070	145	0.90228
			0.04962				**9.3899**
	4	1.764796	50.40	1.1642	1.1510	199	0.90247
			0.06178				**9.3818**

We have found that all of our two stage procedures defined in (2.2)-(2.3) perform remarkably well for many realistic values of C whether C is moderate or large. Table 3.1 corresponds to $C = 43.29$ which may be considered rather on the small side. Yet, the last column from Table 3.1 shows that fairly accurate estimated coverage probabilities tend to exceed our nominal 90% level.

From the $\overline{n}$-column, it is clear that all four procedures tend to oversample on an average compared with C, but this oversampling rate (column 5) goes down steadily as the pilot size m increases. The column 6 provides the *limiting* ratio of the average sample size compared with C. It is remarkable indeed how close the two columns 5 and 6 are even when $C = 43.29$. It is certainly good to know that "asymptotics" kick in this early! However, from column 7, it becomes abundantly clear that the probability distributions of all four stopping variables are heavily right-skewed.

In summary, we feel that the average sample size for the procedure (2.2)-(2.3) based on Gini's mean difference (corresponding to $i = 3$) is slightly closer to that of Stein's original procedure (corresponding to $i = 1$). While the average sample sizes for the procedures based on mean absolute deviation (corresponding to $i = 2$) and range (corresponding to $i = 4$) appear just a shade higher than the average sample size for the procedure based on Gini's mean difference (corresponding to $i = 3$), all three of them are truly very close to each other. If observations from a normal population arrive with some suspect outliers, then one may feel a bit wary to implement Stein's procedure (corresponding to $i = 1$). We have found that in such cases, these other three procedures (corresponding to $i = 2, 3, 4$) hold up better than Stein's original procedure (corresponding to $i = 1$).

4. Known Positive Lower Bound for the Variance

A multi-stage or sequential procedure with its associated stopping variable or final sample size N is customarily called *asymptotically second-order efficient* (Ghosh and Mukhopadhyay, 1981) if $E_{\mu,\sigma}[N - C]$ remains bounded as $d \to 0$. The Stein procedure and the other three two-stage procedures described in the previous section have exact consistency and asymptotic consistency properties, but they do *not* have even the basic asymptotic first-order efficiency property. One may refer to Woodroofe (1977) and Lai and Siegmund (1977,1979), among other sources, for approaches to second-order approximations.

In a situation where one has some preliminary information about a positive lower bound for the population variance σ^2, Mukhopadhyay and Duggan (1997) showed that appropriately modified Stein procedure enjoyed the asymptotic second order efficiency property. A natural question that arises here is: Will analogously modified versions of other stopping rules defined here in (2.2) enjoy first- and second-order efficiency properties? This research is ongoing, and we cannot provide a definitive answer yet, but we feel rather strongly that the answer should probably be in the affirmative.

We should point out that this literature has since grown substantially. A series of solo as well as collaborative communications from M. Aoshima, N. Mukhopadhyay, Y. Takada, K. Yata are especially noteworthy. However, we refrain from giving more specifics in this preliminary note.

Suppose that $\sigma > \sigma_L(> 0)$ and σ_L is known. In this case $C > a^2\sigma_L^2/d^2$ and this lower bound is known! Along the line of Mukhopadhyay and Duggan (1997), we let $m_0(\geq 2)$ be the minimal sample size and define:

$$m \equiv m(d) = \max\left\{m_0, \left\langle a^2\sigma_L^2/d^2\right\rangle + 1\right\}. \tag{4.1}$$

We begin with pilot observations $X_1, ..., X_m$ and define the final sample size:

$$R^{(i)} \equiv R^{(i)}(d) = \max\left\{m, \left\langle b_{m,\alpha}^{(i)2} U_m^{(i)2}/d^2 \right\rangle + 1\right\}, i = 1, 2, 3, 4, \tag{4.2}$$

along the lines of (2.2).

If $R^{(i)} = m$, no more observations are recorded beyond the pilot set, but if $R^{(i)} > m$, then we would record $R^{(i)} - m$ additional observations in the second stage. Finally, based on the combined data $X_1, ..., X_{R^{(i)}}$ from both stages, we would propose the following fixed-width confidence interval

$$I_{R^{(i)}}^{(i)} = \left[\overline{X}_{R^{(i)}} \pm d\right] \tag{4.3}$$

for μ, $i = 1, 2, 3, 4$.

We emphasize that Mukhopadhyay and Duggan's (1997) two-stage procedure developed the final sample size $R^{(1)}$ and the associated fixed-width confidence interval $I_{R^{(1)}}^{(1)}$ for μ.

Table 4.1. Estimated Average Final Sample Size, Maximum Sample Size, and Coverage Probability Under Two-Stage Procedure (4.1)-(4.3): $\alpha = 0.10, a = 1.645, d = 0.5, C = 43.29, m_0 = 5$

m $\sigma_{\mathbf{L}}$	(4.2) index i	$b_{m,\alpha}^{(i)}$	$\overline{n}$ $\mathbf{s}(\overline{n})$	$\overline{n}/C$	$b_{m,\alpha}^{(i)2}a^{-2}$	$\max(n)$	$\overline{p}$ $\mathbf{s}(\overline{p}) \times \mathbf{10}^4$
11	1	1.812461	53.00	1.2243	1.21418	242	0.90304
1			**0.07434**				**9.3573**
	2	1.828029	53.93	1.2458	1.23513	274	0.90192
			0.07993				**9.4053**
	3	1.818508	53.10	1.2266	1.2223	254	0.90226
			0.07529				**9.3908**
	4	1.846661	54.99	1.2703	1.2604	299	0.90100
			0.08570				**9.4445**
30	1	1.697261	46.95	1.0845	1.06519	119	0.90342
1.65			**0.03703**				**9.34089**
	2	1.708158	47.46	1.0963	1.07892	124	0.90383
			0.03971				**9.32316**
	3	1.698744	46.98	1.0852	1.06706	122	0.90331
			0.03744				**9.34564**
	4	1.736417	49.24	1.1374	1.11491	239	0.90676
			0.05051				**9.19491**

Along the line of the data analyses described in Section 3.1, we ran $100,000$ simulations each, implementing all four modified two-stage procedures from (4.1)-(4.3) under many configurations with different choices of $m_0, \sigma_L, \alpha, d, C$ by drawing random samples from a $N(5, 4)$ population. In Table 4.1, we simply highlight two specific illustrations.

It is again noteworthy how close the two columns 5 and 6 are when $C = 43.29$. It is again good to know that "asymptotics" kick in this early! However, from column 7, it becomes abundantly clear that the probability distributions of all four stopping variables are right-skewed, however, the extent of skewness is visibly less now than what we had observed in the case of the procedures (2.2)-(2.3).

Table 4.2. Estimated Values of $E_{\mu,\sigma}[R^{(i)} - C]$ Under
Two-Stage Procedure (4.1)-(4.3):
$\alpha = 0.10, a = 1.645, d = 0.5, C = 43.29, m_0 = 5$

m $\sigma_{\mathbf{L}}$	(4.2) index i	$\overline{r}^{(i)} - C$ values				
11	1	9.755025	9.775675	9.844525	9.875505	9.769575
1		9.703105	9.721545	9.694035	9.837535	9.833515
	2	10.65886	10.63817	10.75354	10.76129	10.68779
		10.60595	10.63025	10.60405	10.76835	10.74475
	3	9.867775	9.875965	9.943805	9.963875	9.876965
		9.795235	9.818285	9.798655	9.954495	9.939645
	4	11.74031	11.86000	11.82571	11.92762	11.74636
		11.70858	11.72324	11.66931	11.83700	11.85849
30	1	3.647325	3.655285	3.651765	3.594345	3.657865
1.65		3.666105	3.587775	3.613645	3.664185	3.665145
	2	4.196405	4.191895	4.219975	4.140235	4.205325
		4.190065	4.117195	4.178455	4.204255	4.194285
	3	3.702345	3.700165	3.710985	3.645085	3.712255
		3.712605	3.631335	3.670615	3.712825	3.708115
	4	5.842975	5.906655	5.824335	5.832855	5.840095
		5.924275	5.851385	5.777935	5.880545	5.935775

From Table 4.1, it is clear that the average oversampling rate has gone down significantly compared with what we had seen in Table 3.1, especially when σ_L was chosen 1.65 (closer to unknown σ) instead of 1. A drop from $22\% - 27\%$ to $8\% - 13\%$!

In order to grasp whether asymptotic second-order properties could possibly prevail, we replicated the process of $100,000$ simulations 10 times each for the two-stage procedures (4.1)-(4.3) under the same scenario as before. In Table 4.2, we exhibit 10 observed values of $\overline{r}^{(i)} - C$ each estimating $E_{\mu,\sigma}[R^{(i)} - C], i = 1, 2, 3, 4$ when $\sigma_L = 1, 1.65$.

We again find that the procedure based on Gini's mean difference (corresponding to $i = 3$) and Stein's original procedure (corresponding to $i = 1$) are very close to each other. While $E_{\mu,\sigma}[R^{(4)} - C]$ for the procedure based on range (corresponding to $i = 4$) appears just a shade higher than $E_{\mu,\sigma}[R^{(2)} - C]$ based on mean absolute deviation (corresponding to $i = 2$), all three of them stay tightly close to each other.

It is clear that overall $\overline{r}^{(i)}$ values are much tighter around C when σ_L was chosen 1.65 (closer to unknown σ) instead of 1.

In summary, we feel strongly that in all likelihood, the three modified procedures from (4.1)-(4.3) corresponding to $i = 2, 3, 4$ are probably asymptotically second-order efficient even though we have not proved such a result yet. More theoretical assessments will be forthcoming.

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Abstract

We revisit fixed-width $(= 2d)$ confidence interval procedures with a preassigned confidence coefficient $(\geq 1 - \alpha)$ for the mean μ of a normal distribution when its variance σ^2 is unknown. Had σ been known, the required optimal fixed sample size would be $C \equiv a^2\sigma^2/d^2$ where $a \equiv a_\alpha$ is the upper $50\alpha\%$ point of $N(0, 1)$. In his fundamental two-stage procedure, Stein (1945,1949) estimated C by replacing σ^2 with a sample variance from pilot data of size $m(\geq 2)$ and a with $t_{m-1,\alpha/2}$. Mukhopadhyay (1982) opened the possibility of incorporating less traditional estimators of σ^2.

In this note, we focus on estimating σ^2 by a statistic U_m^2 defined via *mean absolute deviation* (MAD), *range*, and *Gini's mean difference.* Obviously, then, $t_{m-1,\alpha/2}$ must be replaced by the upper $50\alpha\%$ point corresponding to a pivotal distribution of the sample mean standardized by U_m. This way, we explore the role of Mukhopadhyay's (1982) two-stage confidence interval procedure when the requisite sample size is determined through MAD, range, or Gini's mean difference along with associated exact and first-order properties. Next, we also briefly revisit Mukhopadhyay and Duggan's (1997) updated Stein's two-stage methodology that was proposed when a known positive lower bound σ_L^2 was available for σ^2 in the present light with hints of associated second-order properties. We supplement this ongoing methodological research with data analyses.