Asymptotic properties of U-processes under long-range dependence and applications

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1 Introduction

Since the seminal work of [8], U-statistics have been widely studied to investigate the asymptotic properties of many statistics such as the sample variance, the Gini's mean difference and the Wilcoxon one-sample statistic, see [13] for other examples. One of the most powerful tools used to derive the asymptotic behavior of U-statistics is the Hoeffding's decomposition introduced in [8]. In the i.i.d and weak dependent frameworks, it provides a decomposition of a U-statistic into several terms having different orders of magnitudes, and in general the one with the leading order determines the asymptotic behavior of the U-statistic, see [13, 3] and the references therein for further details. A recent review of the properties of U-statistics in various frameworks is presented in [9]. In the case of processes having a long-range dependent structure, decomposition ideas are also crucial. However, in the case of Gaussian long-memory processes, the classical Hoeffding's decomposition may not provide the complete asymptotic behavior of U-statistics because all terms of this decomposition may contribute to the limit, see for example [5]. In this case, the asymptotic study of U-statistics can be achieved by using an expansion in Hermite polynomials, see [4, 5]. For a large class of processes including linear and nonlinear processes, a new decomposition is discussed in [9]. These authors use martingale-based techniques to establish the asymptotic properties of U-statistics.

A very natural extension of U-statistics is the notion of U-processes which encompasses a wide class of estimators. For example, [3] study the Grassberger-Proccacia estimator which can be used to estimate the correlation dimension. In Section 5 of their work, the authors investigate the asymptotic properties of U-processes when the underlying observations are functionals of an absolutely regular process, that is, short-memory processes. As far as we know, the asymptotic properties of U-processes in the case of long-range dependence setting have not been established yet, and this is the subject of this paper. More precisely, our contribution consists first in extending the results of [3] in order to address the long-range dependence case, second in extending the results obtained in [4] to functions of two variables and third in extending the results of [9] to U-processes. The authors of the latter paper establish the asymptotic properties of U-statistics involving causal but non necessarily Gaussian long-range dependent processes whereas, in our paper, we establish the asymptotic properties of U-processes involving Gaussian long-range dependent processes. The authors in [9] use a martingale decomposition and we use a Hoeffding decomposition or a decomposition in Hermite polynomials.

Consider the U-process defined by

$$(1) \quad U_n(r) = \frac{1}{n(n-1)} \sum_{1 \le i \ne j \le n} \mathbb{1}_{\{G(X_i,X_j) \le r\}} , \quad r \in I$$

where I is an interval included in $\mathbb{R}$, G is a symmetric function *i.e.* $G(x,y) = G(y,x)$ for all x, y in $\mathbb{R}$, and the process $(X_i)_{i \ge 1}$ satisfies the following assumption:

(A1) $(X_i)_{i\ge1}$ is a stationary mean-zero Gaussian process with covariances $\rho(k) = \mathbb{E}(X_1 X_{k+1})$ satisfying:

$$\rho(0) = 1 \text{ and } \rho(k) = k^{-D} L(k), \ 0 < D < 1 ,$$

where L is slowly varying at infinity and is positive for large k.

Note that, for a fixed r, $U_n(r)$ is a U-statistic based on the kernel $h(\cdot, \cdot, r)$ where

$$(2) \quad h(x, y, r) = \mathbb{1}_{\{G(x,y) \le r\}} , \forall x, y \in \mathbb{R} \text{ and } r \in I .$$

We show in this paper that the asymptotic properties of the U-process $U_n(\cdot)$ depends on the value of D and on the Hermite rank m of the class of functions $\{h(\cdot, \cdot, r) - U(r), r \in I\}$, defined in Section 2. We obtain the rate of convergence of $U_n(\cdot)$ and also provide the limiting process when $D > 1/2$, $m = 2$ and when $D < 1/m$, $m = 1, 2$. The convergence rate in the former case is of order $\sqrt{n}$ whereas it is of order $n^{mD/2}/L(n)^{m/2}$ in the latter. These results are stated in Theorems 1 and 2, respectively. They can be applied to derive the asymptotic properties of well-known robust location and scale estimators such as the Hodges-Lehmann estimator defined in [7] and the Shamos scale estimator proposed by [14] and analyzed by [2]. Theorems 1 and 2 allow us to establish novel asymptotic properties of these estimators in the long-range dependence context. The most striking result is that these robust estimators have the same asymptotic distribution as the classical estimators.

The paper is organized as follows. In Section 2, Theorems 1 and 2 are stated. In Section 3, we derive the asymptotic properties of some quantile estimators. Section 4 presents new asymptotic results in the context of long-range dependence. In this section, central and non-central limit theorems are provided for several statistics as an illustration of the theory presented in Sections 2 and 3. These statistics are the Hodges-Lehmann estimator defined in [7], and a robust scale estimator proposed by [14] and [2]. In Section 5, we investigate through numerical experiments the finite-sample properties of the Hodges-Lehmann estimator and illustrate its robustness with respect to the presence of additive outliers.

2 Main results

We start by introducing the terms involved in the Hoeffding's decomposition of [8]. Recall the definition of $U_n(\cdot)$ in (1) and let $U(\cdot)$ be defined as

$$(3) \quad U(r) = \int_{\mathbb{R}^2} h(x, y, r)\varphi(x)\varphi(y)\mathrm{d}x\mathrm{d}y , \text{ for all } r \text{ in } I ,$$

where φ denotes the p.d.f of a standard Gaussian random variable and h is given by (2). For all x in $\mathbb{R}$, and r in I, let us define

$$(4) \quad h_1(x,r) = \int_{\mathbb{R}} h(x,y,r)\varphi(y)\mathrm{d}y \ .$$

The Hoeffding decomposition amounts to expressing, for all r in I, the difference

$$(5) \quad U_n(r) - U(r) = \frac{1}{n(n-1)} \sum_{1\le i\neq j\le n} [h(X_i, X_j, r) - U(r)] \ ,$$

as

$$(6) \quad U_n(r) - U(r) = W_n(r) + R_n(r) \ ,$$

where

$$(7) \quad W_n(r) = \frac{2}{n}\sum_{i=1}^{n} \{h_1(X_i, r) - U(r)\} \ ,$$

and

$$(8) \quad R_n(r) = \frac{1}{n(n-1)} \sum_{1\le i\neq j\le n} \{h(X_i, X_j, r) - h_1(X_i, r) - h_1(X_j, r) + U(r)\} \ .$$

We now define the Hermite rank of the class of functions $\{h(\cdot,\cdot,r) - U(r), r \in I\}$ which plays a crucial role in understanding the asymptotic behavior of the U-process $U_n(\cdot)$. We shall expand the function $(x, y) \mapsto h(x, y, r)$ in a Hermite polynomials basis of $L^2_\varphi(\mathbb{R}^2)$, that is, the L^2 space on $\mathbb{R}^2$ equipped with product standard Gaussian measures. We use Hermite polynomials with leading coefficients equal to one which are: $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x, \ldots$. We get

$$(9) \quad h(x,y,r) = \sum_{p,q\ge 0} \frac{\alpha_{p,q}(r)}{p!q!} H_p(x)H_q(y) \ , \text{ in } L^2_\varphi(\mathbb{R}^2) \ ,$$

where

$$(10) \quad \alpha_{p,q}(r) = \mathbb{E}[h(X,Y,r)H_p(X)H_q(Y)] \ ,$$

and where (X,Y) is a standard Gaussian vector that is X and Y are independent standard Gaussian random variables. Thus,

$$(11) \quad \mathbb{E}[h^2(X,Y,r)] = \sum_{p,q\ge 0} \frac{\alpha^2_{p,q}(r)}{p!q!} \ .$$

Note that $\alpha_{0,0}(r)$ is equal to $U(r)$ for all r, where $U(r)$ is defined in (3). The Hermite rank of $h(\cdot,\cdot,r)$ is the smallest positive integer $m(r)$ such that there exist p and q satisfying $p + q = m(r)$ and $\alpha_{p,q}(r) \neq 0$. Thus, (9) can be rewritten as

$$(12) \quad h(x,y,r) - U(r) = \sum_{\substack{p,q\ge 0 \\ p+q\ge m(r)}} \frac{\alpha_{p,q}(r)}{p!q!} H_p(x)H_q(y) \ , \text{ in } L^2_\varphi(\mathbb{R}^2) \ .$$

The Hermite rank m of the class of functions $\{h(\cdot,\cdot,r) - U(r)\ , r \in I\}$ is the smallest index $m = p + q \geq 1$ such that $\alpha_{p,q}(r) \neq 0$ for at least one r in I, that is, $m = \inf_{r \in I} m(r)$.

By integrating with respect to y in (9), we obtain the expansion in Hermite polynomials of h_1 as a function of x:

$$(13)\quad h_1(x,r) - U(r) = \sum_{p \geq 1} \frac{\alpha_{p,0}(r)}{p!} H_p(x)\ ,\ \text{in } L^2_{\varphi}(\mathbb{R})\ ,$$

where $L^2_{\varphi}(\mathbb{R})$ denotes the L^2 space on $\mathbb{R}$ equipped with the standard Gaussian measure. Let $\tau(r)$ be the smallest integer greater than or equal to 1 such that $\alpha_{\tau,0}(r) \neq 0$, that is, the Hermite rank of the function $h_1(\cdot,r) - U(r)$. The Hermite rank of the class of functions $\{h_1(\cdot,r) - U(r)\ , r \in I\}$ is the smallest index $\tau \geq 1$ such that $\alpha_{\tau,0}(r) \neq 0$ for at least one r. Since $\tau(r) \geq m(r)$, for all r in I, one has

$$(14)\quad \tau \geq m\ .$$

In the sequel, we shall assume that m is equal to 1 or 2. As shown in Section 4, this covers most of the situations of practical interest. Theorem 1, given below, establishes a central limit theorem for the U-process $\{\sqrt{n}(U_n(r) - U(r)), r \in I\}$ when

$$D > 1/m \text{ and } m = 2\ .$$

Theorem 1. *Let I be a compact interval of $\mathbb{R}$. Suppose that the Hermite rank of the class of functions $\{h(\cdot,\cdot,r) - U(r)\ , r \in I\}$ as defined in (12) is $m = 2$ and that Assumption (A1) is satisfied with $D > 1/2$. Assume that h and h_1, defined in (2) and (4), satisfy the three following conditions:*

(i) There exists a positive constant C such that for all s, t in I, u, v in $\mathbb{R}$,

$$(15)\quad \mathbb{E}[|h(X+u, Y+v, s) - h(X+u, Y+v, t)|] \leq C|t-s|\ ,$$

where (X, Y) is a standard Gaussian vector.

(ii) There exists a positive constant C such that for all $k \geq 1$,

$$(16)\quad \mathbb{E}[|h(X_1+u, X_{1+k}+v, t) - h(X_1, X_{1+k}, t)|] \leq C(|u| + |v|)\ ,$$

$$(17)\quad \mathbb{E}[|h(X_1, X_{1+k}, s) - h(X_1, X_{1+k}, t)|] \leq C|t-s|\ .$$

(iii) There exists a positive constant C such that for all t, s in I, and x, u, v in $\mathbb{R}$,

$$(18)\quad |h_1(x+u, t) - h_1(x+v, t)| \leq C(|u| + |v|)\ ,$$

and

$$(19)\quad |h_1(x, s) - h_1(x, t)| \leq C|t-s|\ .$$

Then the U-process

$$\{\sqrt{n}(U_n(r) - U(r)), r \in I\}$$

defined in (1) and (3) converges weakly in the space of cadlag functions $\mathcal{D}(I)$ *equipped with the topology of uniform convergence to the zero mean Gaussian process* $\{W(r), r \in I\}$ *with covariance structure given by*

$$\begin{aligned}(20)\quad \mathbb{E}[W(s)W(t)] = 4\,\mathrm{Cov}(h_1(X_1, s), h_1(X_1, t)) \\ + 4\sum_{\ell\geq 1}\{\mathrm{Cov}(h_1(X_1, s), h_1(X_{\ell+1}, t)) + \mathrm{Cov}(h_1(X_1, t), h_1(X_{\ell+1}, s))\}\ .\end{aligned}$$

The proof of the theorem follows from the decomposition (6) and uses that $\{\sqrt{n}W_n(r), r \in I\}$ converges weakly in the space of cadlag functions $\mathcal{D}(I)$ equipped with the topology of uniform convergence and $\sup_{r\in I}\sqrt{n}|R_n(r)| = o_P(1)$. For a detailed proof, we refer the reader to [10].

When $D < 1/m$, W_n and R_n are not the leading term and the remainder term, respectively. Note that, on one hand, for a fixed r, Corollary 2 of [4] gives $R_n(r) = O_P(n^{-D}L(n))$ for any D in $(0, 1)$. On the other hand, if $D < 1/\tau$, where τ is defined in (14), Theorem 6 of [1] implies that $W_n(r) = O_P(n^{-\tau D/2}L(n)^{\tau/2})$ and if D is in $(1/\tau, 1/m)$, $W_n(r) = O_P(n^{-1/2})$ by Theorem 4 of [1]. Thus, if for instance, $\tau = m = 2$, $W_n(r)$ and $R_n(r)$ may be of the same order $O_P(n^{-D}L(n))$. Hence, to study the case $D < 1/m$, we shall introduce a different decomposition of $U_n(\cdot)$ based on the expansion of h in the basis of Hermite polynomials given by (9). Thus, $U_n(r)$ defined in (1) can be rewritten as follows

$$(21)\quad n(n-1)\{U_n(r) - U(r)\} = \widetilde{W}_n(r) + \widetilde{R}_n(r)\ ,$$

where

$$(22)\quad \widetilde{W}_n(r) = \sum_{1\leq i\neq j\leq n}\ \sum_{\substack{p,q\geq 0\\ p+q\leq m}} \frac{\alpha_{p,q}(r)}{p!q!}H_p(X_i)H_q(X_j)\ .$$

Introduce also the Beta function

$$(23)\quad \mathbf{B}(\alpha, \beta) = \int_0^\infty y^{\alpha-1}(1+y)^{-\alpha-\beta}\mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ , \quad \alpha > 0,\ \beta > 0\ .$$

The limiting processes which appear in the next theorem are the standard fractional Brownian motion (fBm) $(Z_{1,D}(t))_{0\leq t\leq 1}$ and the Rosenblatt process $(Z_{2,D}(t))_{0\leq t\leq 1}$. They are defined through multiple Wiener-Itô integrals by

$$(24)\quad Z_{1,D}(t) = \int_{\mathbb{R}}\left[\int_0^t (u-x)_+^{-(D+1)/2}\mathrm{d}u\right]\mathrm{d}B(x), \quad 0 < D < 1\ ,$$

and

$$(25)\quad Z_{2,D}(t) = \int'_{\mathbb{R}^2}\left[\int_0^t (u-x)_+^{-(D+1)/2}(u-y)_+^{-(D+1)/2}\mathrm{d}u\right]\mathrm{d}B(x)\mathrm{d}B(y), \quad 0 < D < 1/2\ ,$$

where B is the standard Brownian motion, see [6]. The symbol $\int'$ means that the domain of integration excludes the diagonal. Note that $Z_{1,D}$ and $Z_{2,D}$ are dependent but uncorrelated. The following theorem treats the case $D < 1/m$ where $m = 1$ or 2.

Theorem 2. *Let I be a compact interval of $\mathbb{R}$. Suppose that Assumption (A1) holds with $D < 1/m$, where $m = 1$ or 2 is the Hermite rank of the class of functions $\{h(\cdot, \cdot, r) - U(r)\ , r \in I\}$ as defined in (12). Assume the following:*

(i) There exists a positive constant C such that, for all $k \geq 1$ and for all s, t in I,

(26) $\mathbb{E}[|h(X_1, X_{1+k}, s) - h(X_1, X_{1+k}, t)|] \leq C|t - s|$.

(ii) U is a Lipschitz function.

(iii) The function $\widetilde{\Lambda}$ defined, for all s in I, by

(27) $\widetilde{\Lambda}(s) = \mathbb{E}[h(X, Y, s)(|X| + |XY| + |X^2 - 1|)]$,

where X and Y are independent standard Gaussian random variables, is also a Lipschitz function.

Then,

$$\left\{n^{mD/2} L(n)^{-m/2} \left(U_n(r) - U(r)\right); r \in I\right\}$$

converges weakly in the space of cadlag functions $\mathcal{D}(I)$, equipped with the topology of uniform convergence, to

$$\{2\alpha_{1,0}(r) k(D)^{-1/2} Z_{1,D}(1); r \in I\}\ , \quad \text{if } m = 1\ ,$$

and to

$$\{k(D)^{-1}\left[\alpha_{1,1}(r) Z_{1,D}(1)^2 + \alpha_{2,0}(r) Z_{2,D}(1)\right]; r \in I\}\ , \quad \text{if } m = 2\ ,$$

where the fractional Brownian motion $Z_{1,D}(\cdot)$ and the Rosenblatt process $Z_{2,D}(\cdot)$ are defined in (24) and (25) respectively and where

(28) $k(D) = \mathbf{B}((1 - D)/2, D)$,

where $\mathbf{B}$ is the Beta function defined in (23).

For a detailed proof, we refer the reader to [10].

3 Asymptotic behavior of empirical quantiles

We shall apply Theorems 1 and 2 in the preceding section to empirical quantiles. Recall that if $V : I \longrightarrow [0, 1]$ is a non-decreasing cadlag function, where I is an interval of $\mathbb{R}$, then its generalized inverse V^{-1} is defined by $V^{-1}(p) = \inf\{r \in I,\ V(r) \geq p\}$. This applies to $U_n(r)$ and $U(r)$ since these are non-decreasing functions of r. We derive in the following corollaries the asymptotic behavior of the empirical quantile $U_n^{-1}(\cdot)$ using Theorems 1, 2 and the functional Delta method (Theorem 20.8 in [15]).

Corollary 1. *Let p be a fixed real number in $(0, 1)$. Assume that the conditions of Theorem 1 are satisfied. Suppose also that there exists some r in I such that $U(r) = p$, that U is differentiable at r and that $U'(r)$ is non null. Then, as n tends to infinity,*

$$\sqrt{n}(U_n^{-1}(p) - U^{-1}(p)) \xrightarrow{d} -W(U^{-1}(p))/U'(U^{-1}(p))\ ,$$

where W is a Gaussian process having a covariance structure given by (20).

Corollary 2. *Let p be a fixed real number in $(0,1)$. Assume that the conditions of Theorem 2 are satisfied. Suppose also that there exists some r in I such that $U(r) = p$, that U is differentiable at r and that $U'(r)$ is non null. Then, as n tends to infinity,*

$$\frac{n^{mD/2}}{L(n)^{m/2}}(U_n^{-1}(p) - U^{-1}(p))$$

converges in distribution to

$$-2k(D)^{-1/2}\frac{\alpha_{1,0}(U^{-1}(p))}{U'(U^{-1}(p))}Z_{1,D}(1) \ , \ \text{if } m = 1 \ ,$$

and to

$$-k(D)^{-1}\left\{\alpha_{1,1}(U^{-1}(p))Z_{1,D}(1)^2 + \alpha_{2,0}(U^{-1}(p))Z_{2,D}(1)\right\}/U'(U^{-1}(p)) \ , \ \text{if } m = 2 \ ,$$

where $Z_{1,D}(\cdot)$ and $Z_{2,D}(\cdot)$ are defined in (24) and (25) respectively, $k(D)$ in (28) and $\alpha_{p,q}(\cdot)$ is defined in (10).

4 Applications

In this section, we explain how the results established in Sections 2 and 3 can be used to study the asymptotic properties of estimators based on U-processes in the long-range dependence setting such as the Hodges-Lehmann estimator and the Shamos scale estimator. Other examples of application can be found in [12] and [10].

4.1 Hodges-Lehmann estimator

Consider the problem of estimating the location parameter of a long-range dependent Gaussian process. Assume that $(Y_i)_{i\geq 1}$ satisfy $Y_i = \theta + X_i$ where $(X_i)_{i\geq 1}$ satisfy Assumption (A1). To estimate the location parameter θ, [7] suggest using the median of the average of all pairs of observations. The statistic they propose is

$$\text{(29)} \quad \widehat{\theta}_{HL} = \text{median}\left\{\frac{Y_i + Y_j}{2}; 1 \leq i < j \leq n\right\} = \theta + \text{median}\left\{\frac{X_i + X_j}{2}; 1 \leq i < j \leq n\right\} .$$

Define the U-process $U_n(r)$, $r \in \mathbb{R}$ by (1), where $G(x,y) = (x+y)/2$. The Hodges-Lehmann estimator may be then expressed as

$$\widehat{\theta}_{HL} = \theta + U_n^{-1}(1/2) .$$

We proved in [10] that the following result holds.

Proposition 1. *Assume that $(Y_i)_{i\geq 1}$ satisfy $Y_i = \theta + X_i$ where $(X_i)_{i\geq 1}$ satisfy Assumption (A1) then the Hodges-Lehmann estimator $\widehat{\theta}_{HL}$ defined in (29) satisfies*

$$n^{D/2}L(n)^{-1/2}(\widehat{\theta}_{HL} - \theta) \xrightarrow{d} \mathcal{N}(0, 2(-D+1)^{-1}(-D+2)^{-1}) ,$$

and has the same asymptotic behavior as the sample mean $\bar{Y}_n = n^{-1}\sum_{i=1}^n Y_i$.

4.2 Shamos scale estimator

Assume that $(Y_i)_{i\geq 1}$ satisfy $Y_i = \sigma X_i$ where $(X_i)_{i\geq 1}$ satisfy Assumption (A1). The results of the previous section can be used to derive the properties of the estimator of the scale σ proposed by [14] and [2]. From $Y_1, \dots, Y_n$, it is defined by

$$(30) \quad \widehat{\sigma}_{SB} = c \,\text{median}\{|Y_i - Y_j|; 1 \leq i < j \leq n\} = c\,\sigma\, \text{median}\{|X_i - X_j|; 1 \leq i < j \leq n\} ,$$

where $c \approx 1.0483$ to achieve consistency for σ in the case of Gaussian distribution. Here $G(x, y) = |x - y|$. The following proposition is proved in [10].

Proposition 2. *Assume that $(Y_i)_{i\geq 1}$ satisfy $Y_i = \sigma X_i$ where $(X_i)_{i\geq 1}$ satisfy Assumption (A1). Then the Shamos-Bickel scale estimator $\widehat{\sigma}_{SB}$ defined in (30) satisfies:*

(i) If $1/2 < D < 1$,

$$\sqrt{n}(\widehat{\sigma}_{SB} - \sigma) \xrightarrow{d} \mathcal{N}(0, \bar{\sigma}^2) , \text{ as } n \to \infty ,$$

where

$$\bar{\sigma}^2 = \frac{2c^2\sigma^2}{\varphi^2(1/(c\sqrt{2}))} \left[\mathrm{Var}(h_1(Y_1/\sigma, 1/c)) + 2 \sum_{k\geq 1} \mathrm{Cov}(h_1(Y_1/\sigma, 1/c), h_1(Y_{k+1}/\sigma, 1/c)) \right]$$

and h_1 is given by

$$h_1(x, r) = \int_{\mathbb{R}} \mathbb{1}_{\{|x-y|\leq r\}} \varphi(y) \mathrm{d}y = \Phi(x + r) - \Phi(x - r) ,$$

Φ and φ being the c.d.f and the p.d.f of a standard Gaussian random variable, respectively.

(ii) If $0 < D < 1/2$,

$$k(D)n^D L(n)^{-1}(\widehat{\sigma}_{SB} - \sigma) \xrightarrow{d} \frac{\sigma}{2}(Z_{2,D}(1) - Z_{1,D}(1)^2) , \text{ as } n \to \infty ,$$

where $k(D)$ is defined in (28) and the processes $Z_{1,D}(\cdot)$ and $Z_{2,D}(\cdot)$ are defined in (24) and (25). The square root of the sample variance estimator $(\sum_{i=1}^n (Y_i - \bar{Y}_n)^2/(n - 1))^{1/2}$ has, in this case, the same asymptotic behavior as $\widehat{\sigma}_{SB}$.

Remark 1. We proved in [11] that in the case (i) of Proposition 2, the asymptotic relative efficiency of $\widehat{\sigma}_{SB}$ with respect to the square root of the sample variance estimator is larger than 86.31%.

5 Numerical experiments

In this section, we investigate the robustness properties of the Hodges-Lehmann estimator defined in Section 4 using Monte Carlo experiments. We shall regard the observations X_t,

$t = 1, \ldots, n$, as a stationary series Y_t, $t = 1, \ldots, n$, corrupted by additive outliers of magnitude ω. Thus we set

(31) $X_t = Y_t + \omega W_t$,

where W_t are i.i.d. Bernoulli$(p/2)$ random variables. $(Y_t)_t$ is a stationary time series and it is assumed that Y_t and W_t are independent random variables. The empirical study is based on 5000 independent replications with $n = 600$. We consider the cases where (Y_t) are Gaussian ARFIMA$(1, d, 0)$ processes, that is,

(32) $Y_t = (I - \phi B)^{-1}(I - B)^{-d} Z_t$,

where B denotes the backward operator, $\phi = 0.2$ and $d = 0.1$ corresponding to $D = 0.8$, where D is defined in (A1) and (Z_t) are i.i.d $\mathcal{N}(0, 1)$.

In the sequel, we illustrate the results of Proposition 1. In Figure 1, the empirical density functions of $\widehat{\theta}_{HL}$ and $\bar{X}_n$ are displayed when X_t has no outliers, that is $\omega = 0$ in (31) (left) and when there are some outliers such as $p = 10\%$ and $\omega = 10$ in (31) (right). In the case of no outlier (left part of Figure 1) both shapes are similar to the limit indicated in Proposition 1, that is, a Gaussian density with mean zero. In the presence of outliers in the observations (right part of Figure 1) the sample mean is much more sensitive to the presence of outliers than the Hodges-Lehmann estimator. Further numerical experiments can be found in [12], [11] and [10].

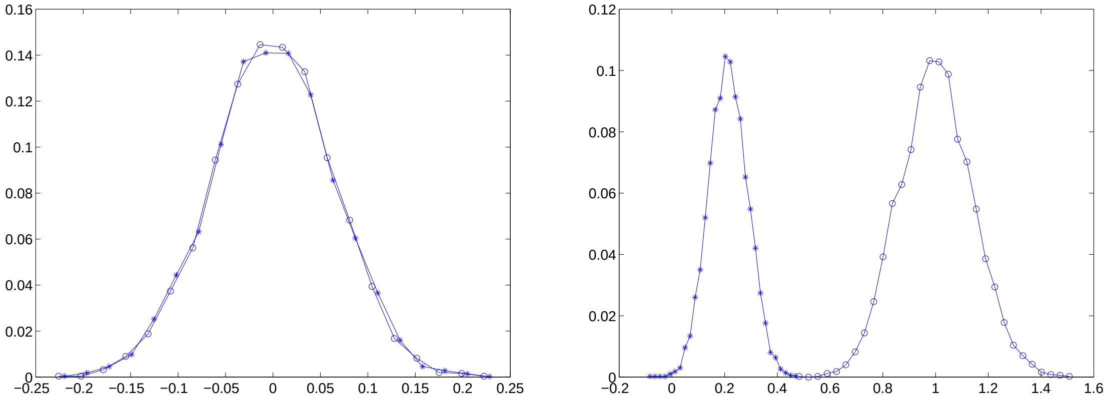

Figure 1: Empirical densities of the quantities $\widehat{\theta}_{HL}$ ('*') and $\bar{X}_n$ ('o') for the ARFIMA$(1, d, 0)$ model with $d = 0.1$ without outliers (left) and with outliers such as $p = 10\%$ and $\omega = 10$ (right).

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