Applications of ambit fields

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Ambit fields (and processes) were introduced by Barndorff-Nielsen and Schmiegel, and applied in turbulence modelling and tumor growth (see Barndorff-Nielsen and Schmiegel (2004,2007) and Barndorff-Nielsen et al. (2004)). Recently, the class of models have found applications to mathematical finance and the theory of stochastic partial differential equations (spde). We present in this short note how ambit fields provide a general class for studying spde's of parabolic type, and how financial models may be recasted in an ambit setting. The presentation is based on the papers Barndorff-Nielsen et al. (2010a,2010b,2011).

Ambit fields

An ambit field is defined for $t \in R$ and x in (some Borel subset of) R^k

$$(1) \qquad Y_t(x) = \mu + \int_{\mathcal{A}_t(x)} g(y,s;x,t)\sigma_s(y)L(dy,ds) + \int_{\mathcal{D}_t(x)} q(y,s;x,t)a_s(y)\,dy\,ds\,,$$

where $\mathcal{A}_t(x)$ and $\mathcal{D}_t(x)$ are ambit sets, that is, measurable subsets of $R^k \times (-\infty, t]$ and μ is a constant. Furthermore, g and q are deterministic functions, and $a_t(x)$ and $\sigma_t(x)$ are positive random fields (referred to as the intermittency fields in turbulence applications). Finally, L is a so-called *Lévy basis*, which means that L is an independently scattered random measure on R^{k+1} for which the law is infinitely divisible.

To make the definition of Y precise, one must define the stochastic integral with respect to L and provide conditions on the integrand field $(s,y) \mapsto g(y,s;x,t)\sigma_s(y)$. This is done in Barndorff-Nielsen et al. (2011). The integrand field $(s,y) \mapsto q(y,s;x,t)a_s(y)$ must be Lebesgue integrable to make the last term in $Y_t(x)$ well-defined.

Energy and weather markets

In energy markets like for instance electricity or gas, one is trading in forward and futures contracts which deliver the underlying commodity over a contracted period of time. For example, one may in the German power market EEX buy or sell electricity for given months or quarters in the future. In the weather markets, one has a similar structure, where futures are written on a weather index (temperature, say), measured over a period like a month or a season.

Denoting the price of a forward on electricity as $F(t,T_1,T_2)$, with $t \le T_1 < T_2$ and $[T_1,T_2]$ being the delivery period, one may by no-arbitrage arguments (see Benth et al. (2008)) derive

$$(2) \qquad F(t,T_1,T_2) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} f(t,T)\,dT\,.$$

Here, $f(t,T) := F(t,T,T)$, that is, the forward price of a contract with instantaneous delivery of power at time T which of course is non-existent in the market.[1] Forwards on gas may be represented in a similar fashion.

[1] Note that the reason for taking the average of $f(t,T)$ is that the market by convention denote forward prices per MWh

The forward price of a contract on an index like the cumulative temperature over a period $[T_1, T_2]$ may be written as (see Benth et al. (2008))

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} f(t, T)\, dT\,, \tag{3}$$

with $f(t, T) := F(t, T, T)$, the forward price of instantaneous "delivery of temperature"[2] at time T. Forwards on temperatures are traded at the Chicago Mercantile Exchange.

To model energy and weather forwards, one may reduce this to specify a stochastic dynamics of $f(t, T)$. Changing to the Musiela parametrization, that is, letting $x = T - t$, we consider the random field $g(t, x) := f(t, t + x)$, $x \geq 0$. In the risk-neutral framework, the forward price process $t \mapsto f(t, T)$ has to be a martingale, which translate into a dynamics for g given by the Musiela stochastic partial differential equation (for artithmetic models of g driven by a Brownian motion on R)

$$dg(t, x) = \frac{\partial g}{\partial x}(t, x)\, dt + \sigma(t, x)\, dB(t)\,. \tag{4}$$

These models have been adopted for fixed-income theory, where g is associated with the Heath-Jarrow-Morton approach to modelling forward rates. A mild solution to this spde is

$$g(t, x) = S_t g(0, x) + \int_0^t S_{t-s}\sigma(s, x)\, dB(s)\,. \tag{5}$$

Here, S_y is the right-shift operator, that is $S_y\sigma(s, x) = \sigma(s, x+y)$, being the semigroup with generator $\partial/\partial x$. As we see, this is a special case of an ambit field. For an extensive analysis of this, and generalizations provided by the ambit fields, see Barndorff-Nielsen et al. (2010b).

Stochastic partial differential equations

Consider the so-called cable equation, being the solution $v(t, x)$ of the spde

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v + \dot{W} \tag{6}$$

for $0 < x < K, t > 0$, $v(0, x) = v_0(x)$ and Dirichlet boundary conditions

$$\frac{\partial v}{\partial x}(t, 0) = \frac{\partial v}{\partial x}(t, K) = 0\,. \tag{7}$$

Here, $\dot{W}$ is *white noise* in time and space. Under some conditions on $v_0(x)$, the initial condition, a weak solution of v is given by

$$v(t, x) = \int_0^K G_t(x, y)v_0(y)\, dy + \int_0^t \int_0^K G_{t-s}(x, y)W(dy, ds)\,. \tag{8}$$

Here, W is a Gaussian Lévy basis, with $\dot{W}$ referring to its noise, and G is the Green's function. We observe that $v(t, x)$ is of the form of an ambit field. Ambit fields may provide a source of models which can give mild solutions to the cable equation for more general noise terms than $\dot{W}$, including for example the addition of volatility and coloured noise $\sigma_t(x)\dot{L}$. Since the cable equation models diffusion of heat in a row, for example, this may be relevant as an extension to better describe the randomness in the medium the heat is transported, say. We refer to Barndorff-Nielsen et al. (2011) for details on the theory of spde's and ambit fields.

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