Comparison of Approximations for Compound Poisson Processes

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Introduction

The aim of this paper[1] is to provide a comparison of the error in several approximation methods for the cumulative aggregate claim distribution customarily used in the collective model of insurance theory (see e.g. Cramér, 1955, or Beard et al., 1990). In this theory it is usually supposed that a portfolio is at risk for a time period of length t. The claims take place according to a Poisson process of intensity μ, so that the number of claims in $[0,t]$ is a Poisson random variable N with parameter $\lambda = \mu t$. Each single claim is a random variable X_i for $i = 1, \ldots, N$ with a common distribution. We consider the random sum $S_N = \sum_{i=1}^{N} X_i$, i.e. a compound Poisson process representing the aggregate claim or total claim amount process in $[0,t]$.

We denote as μ_i the $i-$th noncentral moment of X_i. Then the aggregate claim process has moments:

$$\mathsf{E} S_N = \mu_1 \lambda, \qquad \mathsf{V}(S_N) = \mu_2 \lambda.$$

We will write $\gamma_1 := \frac{\mu_3}{\mu_2^{3/2}\lambda^{1/2}}$ (skewness index), $\gamma_2 := \frac{\mu_4}{\mu_2^2 \lambda}$ (kurtosis index) and $\gamma_3 := \frac{\mu_5}{\mu_2^{5/2}\lambda^{3/2}}$. Then let $S_N^\star := \frac{S_N - \mathsf{E} S_N}{\sqrt{\mathsf{V}(S_N)}}$. We evaluate the accuracy of nine approximations available in the literature to:

$$F(x) := \mathsf{P}\left\{S_N^\star \leq x\right\},$$

as the Poisson intensity diverges to infinity, i.e. as $\lambda \to \infty$. We consider the difference between the true distribution and the approximating one and we propose to use expansions of this difference related to Edgeworth series to measure the accuracy of the approximation. In order to do so, we will need the Hermite polynomials $He_j(x)$ for $j = 0, 1, \ldots$. The first Hermite polynomials, in the formulation customarily used in Statistics, are given by the formulas $He_0(x) = 1$, $He_1(x) = x$, $He_2(x) = x^2 - 1$ and $He_3(x) = x^3 - 3x$.

Edgeworth Expansion

The following Edgeworth expansion for compound Poisson processes (see e.g. Cramér, 1955) will be used in the following.

Theorem. Consider a compound Poisson process with intensity λ. Let μ_j's be the noncentral moments of the random variable X, whose characteristic function is such that $\lim_{|t|\to\infty} \sup |\phi(t)| < 1$.

[1]The present short paper is based on a longer one by the same authors in which several more methods are considered. The original paper contains full proofs of the results as well as more complete historical accounts concerning their introduction in Insurance.

Then, if $F(x) = \mathsf{P}\left\{\frac{S_N - \mathsf{E}S_N}{\sqrt{\mathsf{V}(S_N)}} \leq x\right\}$:

$$\begin{aligned} F(x) &= \Phi(x) + \phi(x) \cdot \left\{ -\lambda^{-1/2} \cdot \frac{\mu_3}{6\mu_2^{3/2}} \cdot He_2(x) - \lambda^{-1} \cdot \left[\frac{\mu_4}{24\mu_2^2} \cdot He_3(x) + \frac{\mu_3^2}{72\mu_2^{3/2}} \cdot He_5(x) \right] \right. \\ & \left. -\lambda^{-\frac{3}{2}} \cdot \left[\frac{\mu_5}{120\mu_2^{\frac{5}{2}}} \cdot He_4(x) + \frac{\mu_3\mu_4}{144\mu_2^{\frac{7}{2}}} \cdot He_6(x) + \frac{\mu_3^3}{1296\mu_2^{\frac{9}{2}}} \cdot He_8(x) \right] \right\} + o\left(\lambda^{-3/2}\right) \end{aligned}$$

where the remainder term is uniform.

Approximations

We provide expansions for nine approximations, namely Normal, Edgeworth, NP2, NP2a, Adjusted NP2, NP3, Gamma, Inverse Gaussian and Gamma-IG. All the following expansions are available to order $O\left(\lambda^{-3/2}\right)$, but for simplicity of exposition we limit ourselves to $O\left(\lambda^{-1}\right)$ when sufficient.

Normal Approximation. The normal approximation, whose credit goes to F. Lundberg in the first years of the 20th century, is based on an application of the Rényi's version of Anscombe central limit theorem:

$$\begin{aligned} F(x) - \Phi(x) &= \phi(x) \cdot \frac{\gamma_1}{6} \cdot \left(1 - x^2\right) \\ & + x\phi(x) \cdot \left[\frac{\gamma_2}{24} \cdot \left(3 - x^2\right) + \frac{\gamma_1^2}{72} \cdot \left(10x^2 - x^4 - 15\right) \right] + o\left(\lambda^{-1}\right). \end{aligned}$$

Edgeworth Approximation. This approximation is due to Cramér (1955). In this case the error is:

$$\begin{aligned} & F(x) - \Phi(x) - \phi(x) \cdot \frac{\gamma_1}{6} \cdot \left(1 - x^2\right) \\ = \; & x\phi(x) \cdot \left[\frac{\gamma_2}{24} \cdot \left(3 - x^2\right) + \frac{\gamma_1^2}{72} \cdot \left(10x^2 - x^4 - 15\right) \right] + o\left(\lambda^{-1}\right). \end{aligned}$$

Both normal and Edgeworth expansions are good only for very large values of λ.

NP2 Approximation. The NP2 approximation, whose introduction in Insurance is usually credited to Kauppi and Ojantakanen (1969), yields:

$$\begin{aligned} & F(x) - \Phi\left(\frac{3}{\gamma_1} \left[\sqrt{1 + \frac{\gamma_1^2}{9} + \frac{2\gamma_1}{3} \cdot x} - 1 \right] \right) \\ = \; & x\phi(x) \left\{ \frac{\gamma_2}{24} \cdot \left(3 - x^2\right) + \frac{\gamma_1^2}{36} \cdot \left(2x^2 - 5\right) \right\} + o\left(\lambda^{-1}\right). \end{aligned}$$

This approximation is better for the tail when $\frac{\gamma_1^2}{3} - \frac{\gamma_2}{4}$ is near to zero.

NP2a Approximation. This is known in Statistics as first-order "normalizing" Cornish-Fisher expansion; its introduction in Insurance seems to be due to Pentikäinen (1977). We get:

$$F(x) - \Phi\left(x - \frac{\gamma_1}{6} \cdot \left(x^2 - 1\right)\right) = x\phi(x) \left\{ \frac{\gamma_2}{24} \cdot \left(3 - x^2\right) + \frac{\gamma_1^2}{36} \cdot \left(4x^2 - 7\right) \right\} + o\left(\lambda^{-1}\right).$$

Despite the similarity of the formulas, this approximation is by far worse than the previous one for large x: when $\gamma_1 > 0$ and $x \to \infty$, $F(x)$ converges to 1 while $\Phi\left(x - \frac{\gamma_1}{6} \cdot \left(x^2 - 1\right)\right)$ converges to 0.

Adjusted NP2 Approximation. This approximation, due to Ramsay (1991), is based on the computation of b_0, that is the unique root of equation $\gamma_1 = 6b - 4b^3$ lying in the interval $\left[0, 1/\sqrt{2}\right]$, and of $a_0 = \sqrt{1 - 2b_0^2}$. Then the approximation error turns out ot be:

$$F(x) - \Phi\left(-\frac{a_0}{2b_0} + \sqrt{1 + \frac{1}{b_0} \cdot x + \frac{a_0^2}{4b_0^2}}\right) = x\phi(x) \cdot \left(\frac{\gamma_2}{24} - \frac{\gamma_1^2}{18}\right) \cdot \left(3 - x^2\right) + o\left(\lambda^{-1}\right).$$

NP3 Approximation. This approximation is due to Kauppi and Ojantakanen (1969). Let $y(x)$ be the value y, function of x, solving the cubic equation $x = y + \frac{\gamma_1}{6} \cdot (y^2 - 1) + \frac{\gamma_2}{24} \cdot (y^3 - 3y) - \frac{\gamma_1^2}{36} \cdot (2y^3 - 5y)$. Then:

$$\begin{aligned} & F(x) - \Phi(y(x)) \\ = \; & \phi(x)\left\{\frac{\gamma_3}{120} \cdot \left(-x^4 + 6x^2 - 3\right) + \frac{\gamma_1\gamma_2}{24} \cdot \left(x^4 - 5x^2 + 2\right)\right\} \\ & +\phi(x)\left\{\frac{\gamma_1^3}{324} \cdot \left(-12x^4 - 67x^2 - 17\right)\right\} + o\left(\lambda^{-3/2}\right). \end{aligned}$$

Gamma Approximation. The idea, apparently due to Bohman and Esscher (1963) and later revived and popularized by Seal (1977), is to approximate the centered and normalized sum using the random variable $\frac{\gamma_1}{4}\left(\chi^2_{\frac{8}{\gamma_1^2}} - \frac{8}{\gamma_1^2}\right)$, where χ^2_n is a shortcut for $\chi^2_n \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ with positive real n:

$$\begin{aligned} & F(x) - \mathsf{P}\left\{\frac{\gamma_1}{4}\left(\chi^2_{\frac{8}{\gamma_1^2}} - \frac{8}{\gamma_1^2}\right) \le x\right\} \\ = \; & \phi(x) \cdot \left\{\frac{1}{8}\left(\frac{\gamma_1^2}{2} - \frac{\gamma_2}{3}\right) \cdot He_3(x) + \frac{1}{40}\left(\gamma_1^3 - \frac{\gamma_3}{3}\right) \cdot He_4(x)\right\} \\ & +\phi(x) \cdot \left\{\frac{\gamma_1}{48}\left(\frac{\gamma_1^2}{2} - \frac{\gamma_2}{3}\right) \cdot He_6(x)\right\} + o\left(\lambda^{-3/2}\right). \end{aligned}$$

Inverse Gaussian Approximation. As in Chaubey et al. (1998), take $m = \frac{3\mu_2^2\lambda}{\mu_3}$ and $b = \frac{\mu_3}{3\mu_2}$. Therefore, if IG denotes an inverse Gaussian random variable, we get:

$$\begin{aligned} & F(x) - \mathsf{P}\left\{\frac{IG(m,b) - m}{\sqrt{mb}} \le x\right\} \\ = \; & \phi(x) \cdot \left(\frac{5\gamma_1^2 - 3\gamma_2}{432}\right) \cdot \left(6 \cdot He_3(x) + \gamma_1 \cdot He_6(x)\right) \\ & +\phi(x) \cdot \left(\frac{7\gamma_1^3}{216} - \frac{\gamma_3}{120}\right) \cdot He_4(x) + o\left(\lambda^{-3/2}\right). \end{aligned}$$

Our computations show that, according to the second order Edgeworth expansion, the Inverse Gaussian and the Gamma approximations are of comparable accuracy.

Gamma-IG approximation. A further approximation (Chaubey et al., 1998) is obtained as a linear combination of the Gamma and the IG approximation given above. The idea is to use:

$$\begin{aligned} & F(x) - w\mathsf{P}\left\{\frac{\gamma_1}{4}\left(\chi^2_{\frac{8}{\gamma_1^2}} - \frac{8}{\gamma_1^2}\right) \le x\right\} - (1 - w)\,\mathsf{P}\left\{\frac{IG(m,b) - m}{\sqrt{mb}} \le x\right\} \\ = \; & \phi(x) \cdot \left(-\frac{\gamma_1^3}{24} - \frac{\gamma_3}{120} + \frac{6\gamma_2\gamma_1}{135}\right) \cdot He_4(x) + o\left(\lambda^{-3/2}\right) \end{aligned}$$

where $w = \frac{\gamma_2 - \frac{5\gamma_1^2}{3}}{\frac{3\gamma_1^2}{2} - \frac{5\gamma_1^2}{3}} = \frac{10\gamma_1^2 - 6\gamma_2}{\gamma_1^2}$. The error is uniformly $o\left(\lambda^{-1}\right)$.

Computations

Let X be a Gamma random variable with scale parameter 1 and shape parameter 2. We take $\lambda = 10$. Thus $\gamma_1 = 0.5163978$, $\gamma_2 = 0.3333333$ and $\gamma_3 = 0.2581989$. A first striking fact that was not evident from the formulas (but is reliably reproduced by their graphs) is the difference between NP2, NP2a and Adjusted NP2: NP2 and Adjusted NP2, despite the similarity in the formulas, have often very different behaviors and NP2a is reliable only for very small values of γ_1 and not too large values of x. A fact that was expected from the formulas is the similarity between the Gamma and the IG approximations, as is the extreme precision of the Gamma-IG approximation.

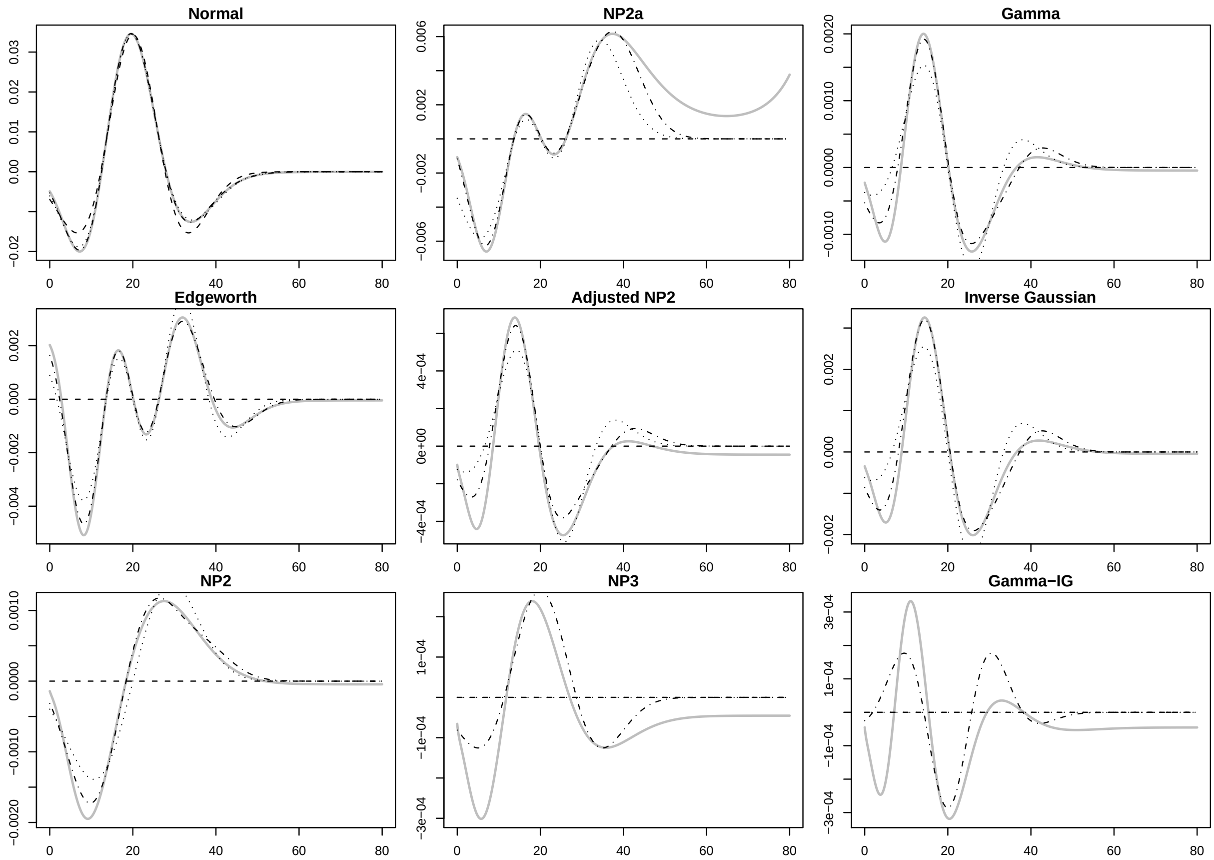

Figure 1: Error of the approximations (grey line) and proposed expansions (dashed line to order $O\left(\lambda^{-1/2}\right)$, dotted line to order $O\left(\lambda^{-1}\right)$, dash-dot line to order $O\left(\lambda^{-3/2}\right)$) for $\lambda = 10$ and X Gamma with scale parameter 1 and shape parameter 2.

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RÉSUMÉ (ABSTRACT)

Nous comparons la précision de neuf méthodes d'approximation différentes pour la distribution agrégée des sinistres habituellement utilisée dans le modèle collectif de théorie de l'assurance. La distance entre la vraie distribution et son approximation est développée en série de puissances de $\lambda^{-1/2}$ *jusqu'à l'ordre* $O\left(\lambda^{-3/2}\right)$*. Pour obtenir ce résultat, nous avons à plusieurs reprises recours à des expansions d'Edgeworth, pour les sommes de Poisson généralisées et les sommes classiques, et à des séries de Taylor. La précision des résultats est évaluée empiriquement.*

We compare the accuracy of nine alternative approximation methods for the cumulative aggregate claim distribution customarily used in the collective model of insurance theory. The distance between the true distribution and the approximation is expanded in a series of powers of $\lambda^{-1/2}$ *up to order* $O\left(\lambda^{-3/2}\right)$*. To achieve this result we use repeatedly Edgeworth expansions for compound Poisson sums and for classical sums, and Taylor series. The accuracy of the results is evaluated empirically.*