On Computing Approximation of Correlations using Bernstein Copula and Probabilistic Tools

Abegaz, Fentaw
Addis Ababa University, Department of Statistics,
P.O.Box 1176, Addis Ababa,Ethopia
E-mail: fentawabegaz@yahoo.com

Mandrekar, V.
Michigan State University, Department of Statistics and Probability
East Lansing,MI 48824, USA
E-mail:mandrekar@stt.msu.edu

Naik-Nimbalkar, Uttara
University of Pune, Department of Statistics
Pune 411007,India
E-mail: uvnaik@stats.unipune.ac.in

Introduction

In insurance problems, value at risk (VaR) is considered one of the most popular risk measure. When VaR is known for each risk, it becomes important to manage VaR of joint position resulting from combination of different dependent risks (e.g. market and credit risks). Here value at risk being the quantile, it is important to know the dependence structure of the risks in order to find the VaR of functions of these risks. Embrechts and his collaborators (2002) have shown that linear correlation is insufficient as a measure of dependence for studying VaR across a wide range of portfolio structures. Also see Embrechts et al (2003a) for a review of the problem and related work. The problem mentioned here was first attacked by W. Hoeffding in 1941 (see collected works of Wassily Hoeffding (Fisher and Sen, 1994) for English version) using what is now known as copula (Sklar(1959), Nelsen (1999)). Hoeffding studies different scale invariant measures of dependence using approximation to copula by a finite series in Legendre polynomials. He, in fact, ends up using insurance data to compute certain correlations (e.g. Kendall's Tau and Spearman's rank correlation). Recently financial practitioners are considering another risk measure called the Expected Shortfall (ES), first proposed by Artzner et al. (1997) and which is same as the measure 'Conditional Value-at-risk'.

Olsen et al (1996) and Li et al (1998) have intoduced the approximations of copulas via Markov operators. In this paper we restrict to approximations by Bernstein copula and follow the ideas of Sancetta and Satchell (2004) and Durrleman et al (2001). We give the approximation using elementary probabilistic tools as in Feller (1971) generalizing the work in Gzyl and Palacios (2003). Further, we establish that the empirical Bernstein copula converges uniformly to the true copula. We note that the measures of dependence, Kendall's Tau, Spearman's rank correlation, Hoeffding's dependence index and Pearson's coefficient of mean square contingency can be approximated by those of approximating Bernstein copula and the empirical Bernstein copula. Based on the approximating (empirical) Bernstein copula, we give a numerical procedure to determine the VaR and ES. These approximations may be helpful as it is known that the biases in VaR estimates are due to the misspecification of the copula. Finally we compute nonparametric estimates of Kendall's tau and the Spearman's rank correlations between well known international indexes. It is observed that sample versions of these two measures are close to the approximations based on the empirical Bernstein copula when the data consist of monthly closing values. However when the data consist of bimonthly closing values, the estimates of Spearman's rank correlations based on the empirical Bernstein copula are closer to the above estimates than the sample versions. The approximation by the Bernstein copula goes through for more than two dimensions.

Bernstein Copula Approximations and Convergences

Let $C(.,.)$ be a two-dimensional copula; that is, it is a function from $[0,1]^2$ to $[0,1]$ satisfying (i) C is nondecreasing in each argument and (ii) $C(u,1)=u$, $C(1,v)=v$ and $C(u,0)=0=C(0,v)$. Let $S_{m,u}$ and $S_{m,v}$ be two independent binomial random variables with parameters (m,u) and (m,v), respectively. Let $E[U]$ denote the expected value of the random variable U. An approximation of the copula function $C(u,v)$ using the Bernstein polynomials of degree m in two variables is given by

$$(1) \qquad (B_mC)(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{m} C\left(\frac{i}{m},\frac{j}{m}\right)P_{i,u}(u)P_{i,v}(v),$$

where $P_{i,u}(u)=\binom{m}{i}u^i(1-u)^{m-i}$, $i=0,1,\cdots,m$. The functional approximation is called the Bernstein copula (Sancetta and Satchell, 2004). An important representation of the Bernstein copula (1) is to express it in terms of expectation taken with respect to two independent binomial distributions as

$$(B_mC)(u,v)=E\left[C\left(\frac{S_{m,u}}{m},\frac{S_{m,v}}{m}\right)\right].$$

Such a representation allows us to deal with an extension of the uniform convergence result in Feller (1971, Theorem 1, p.221) to Bernstein polynomials in two or higher dimensions. We next state a theorem on uniform convergence of the Bernstein copula.

Theorem 1 : *The Bernstein copula $(B_mC)(u,v)$ tends uniformly to $C(u,v)$ as $m\to\infty$.*

Proof: Consider

$$(2) \qquad |(B_mC)(u,v)-C(u,v)|=\left|E\left[C\left(\frac{S_{m,u}}{m},\frac{S_{m,v}}{m}\right)\right]-C(u,v)\right|\le E\left|C\left(\frac{S_{m,u}}{m},\frac{S_{m,v}}{m}\right)-C(u,v)\right|.$$

Let $A_m=C\left(\frac{S_{m,u}}{m},\frac{S_{m,v}}{m}\right)-C(u,v)$ and

$$R_1=\{|S_{m,u}/m-u|\le\delta,|S_{m,v}/m-v|\le\delta\},\ R_2=\{|S_{m,u}/m-u|\le\delta,|S_{m,v}/m,-v|>\delta\}$$
$$R_3=\{|S_{m,u}/m-u|>\delta,|S_{m,v}/m-v|\le\delta\},\ R_4=\{|S_{m,u}/m-u|>\delta,|S_{m,v}/m-v|>\delta\},$$

and let I_R denote the indicator function of the set R. Then from (2),

$$(3) \qquad |(B_mC)(u,v)-C(u,v)|\le E|A_mI_{R_1}|+E|A_mI_{R_2}|+E|A_mI_{R_3}|+E|A_mI_{R_4}|.$$

Since C is uniformly continuous (Nelsen, 1999, Theorem 2.2.4 p.9) given ϵ we can choose a δ such that on R_1, $|C(S_{m,u}/m,S_{m,v}/m)-C(u,v)|<\epsilon$. Thus the first term in (3) is

$$(4) \qquad E|A_mI_{R_1}|<\epsilon E[I_{R_1}]<\epsilon.$$

Since C is bounded, there exists a finite number K such that the second term on the RHS of (3) is bounded above by

$$\begin{aligned} E|A_mI_{R_2}|\le 2KE|I_{R_2}| &\le 2KP[|S_{m,u}/m-u|\le\delta,|S_{m,v}/m-v|>\delta] \\ &\le 2KP[|S_{m,v}-mv|>m\delta]. \end{aligned}$$

But by Hoeffding's inequality (Hoeffding (1963)), $P[|S_{m,v} - mv| > m\delta] \leq 2exp(-2m\delta^2)$. Thus for sufficiently large m,

(5) $$E|A_m I_{R_2}| \leq 4Kexp(-2m\delta^2) \leq \epsilon.$$

Similarly, for sufficiently large m,

(6) $$E|A_m I_{R_3}| \leq 4Kexp(-2m\delta^2) \leq \epsilon \text{ and } E|A_m I_{R_4}| \leq 4Kexp(-2m\delta^2) \leq \epsilon.$$

From (4), (5),(6) and (3), we obtain that given ϵ there exists an m_0 such that for all $m \geq m_0$,

$$\sup_{u,v} |(B_m C)(u, v) - C(u, v)| < 4\epsilon.$$

Since ϵ is arbitrary we get the required result.

Remark 1: Using the above Binomial random variables, it can also be established that if the copula has continuous p-th order partial derivatives, then the partial derivatives of order $\leq p$ of the approximating Bernstein copula converge uniformly to the corresponding partial derivatives of the copula (Lojasiewicz (1988)).

Empirical Bernstein Copula

Let $(U_1, V_1), (U_2, V_2), \cdots, (U_n, V_n)$ be a random sample from a continuous distribution. Define the joint empirical distribution function

$$H_n(u, v) = \frac{1}{n} \sum_{k=1}^{n} I[U_k \leq u, V_k \leq v],$$

and let $F_n(u) = H_n(u, \infty)$ and $G_n(v) = H_n(\infty, v)$ be its associated marginal distributions. We define the empirical copula function C_n by

$$C_n(u, v) = H_n(F_n^{-1}(u), G_n^{-1}(u)),$$

where $F^{-1}(u) = inf\{t \in R | F(t) \geq u\}, \quad 0 \leq u \leq 1.$

The corresponding empirical Bernstein copula is given by

(7) $$(B_m C_n)(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{m} C_n \left(\frac{i}{m}, \frac{j}{m}\right) P_{i,u}(u) P_{i,v}(v).$$

Since the binomial r.v.s $S_{m.u}$ and $S_{m,v}$ are independent of the data, given the data, the empirical Bernstein copula can be expressed in terms of expectation with respect to the binomial distributions as $(B_m C_n)(u, v) = E\left[C_n\left(S_{m,u}/m, S_{m,v}/m\right)\right].$

Theorem 2 *The empirical Bernstein copula converges uniformly to the true copula. That is, as* $m \to \infty$ *and* $n \to \infty$,

$$\sup_{u,v} |(B_m C_n)(u, v) - C(u, v)| \to 0, \ (a.s.).$$

Proof: We note that

(8) $$\sup_{u,v} |(B_m C_n)(u, v) - C(u, v)| \leq \sup_{u,v} |(B_m C_n)(u, v) - (B_m C)(u, v)| + \sup_{u,v} |(B_m C)(u, v) - C(u, v)|.$$

From the Theorem 1 above, the second term on RHS of (8) goes to zero as $m \to \infty$. The first term can be written as

(9) $$\sup_{u,v} |E[C_n(S_{m,u}/m, S_{m,v}/m) - C(S_{m,u}/m, S_{m,v}/m)]|.$$

Deheuvels (1979, 1981) has proved that the empirical copula converges uniformly to the true copula almost surely. Thus given $\epsilon > 0$ there exists an n_0, which does not depend on u and v such that for all $n \geq n_0$, the expression in (9) is less than $\epsilon, (a.s.)$. This completes the proof.

Let $C_{,1}(u, v) = \frac{\partial C(x,y)}{\partial x}|_{x=u,y=v}$ and $C_{,2}(u, v) = \frac{\partial C(x,y)}{\partial y}|_{x=u,y=v}$. We note that $(B_m C_n)$ is a copula and its first order partial derivatives exist and are continuous.

Remark 2: From the above Theorem 2 and Lemma 3.1 of Olsen et al (1996), we get the result that for all $f \in L^p([0, 1]^2)$, $p \in (1, \infty]$, as $m \to \infty$ and $n \to \infty$, for $k = 1, 2$,

$$\int_0^1 \int_0^1 f(u, v)(B_m C_n)_{,k}(u, v)dudv \to \int_0^1 \int_0^1 f(u, v)C_{,k}(u, v)dudv, \quad (a.s.)$$

Remark 3: Suppose that the copula function C(x, y) has continuous first order partial derivatives and the associated marginal distribution functions are continuous. Then using the above results it can be shown that given $\epsilon > 0$ there exists an m_0 and an n_0 such that for all $n \geq n_0$

$$\sup_{u,v} |B_{m_0}(C_n)_{,1}(u, v) - C_{,1}(u, v)| < \epsilon \text{ and} \sup_{u,v} |B_{m_0}(C_n)_{,2}(u, v) - C_{,2}(u, v)| < \epsilon.$$

Further, from the weak convergence of the empirical copula process (Fermanian et al. (2004)), and the fact that $\sum_{i=0}^m \sum_{j=0}^m |P'_{i,u}(u)P_{i,v}(v)| = O(m)$, it follows that as $m \to \infty$, $n \to \infty$ if $m/\sqrt{n} \to 0$, for $k = 1, 2$,

$$\sup_{u,v} |(B_m C_n)_{,k}(u, v) - C_{,k}(u, v)| \to 0, \ \textit{in probability.}$$

Using the law of iterated logarithm for the empirical copula process Deheuvels(1979), we obtain the following result.

Theorem 3 *Suppose that the copula function C(x, y) has continuous second order partial derivatives and the associated marginal distribution functions are continuous. As* $m \to \infty$, $n \to \infty$, *if* $m(log(logn))^{1/2}/n^{1/2} \to 0$, *then*

$$\sup_{u,v} |(B_m C_n)_{,1}(u, v) - C_{,1}(u, v)| \to 0, \text{ and } \sup_{u,v} |(B_m C_n)_{,2}(u, v) - C_{,2}(u, v| \to 0, \quad (a.s.).$$

Remark 4: If the conditions of the Theorem 3 hold, then the sequence $B_m C_n$ converges to C strongly (Li et al. (1998)). That is, as $m \to \infty$, $n \to \infty$ and $m(log(logn))^{1/2}/n^{1/2} \to 0$,

$$\int_0^1 \left| \int_0^1 ((B_m c_n)(u, v)f(v) - c(u, v)f(v))dv \right| du \to 0, \ (a.s.), \ \textit{for all} \ f \in L^1[0, 1],$$

where $c(u, v) = \frac{\partial^2 C}{\partial u \partial v}$ denotes the density of the copula function C and $(B_m c_n)$ denotes the density of the empirical Bernstein copula.

We give below some measures of associations and measures of risk based on a (unknown) copula C with the required properties that can be approximated by the corresponding measures of a (empirical) Bernstein copula with an appropriate m up to any given degree of accuracy.

Measures of Association or Dependence

Spearman's rank correlation (ρ_s)**:** $\rho_s = 12 \int_0^1 \int_0^1 C(u, v)dudv - 3$,
Kendall's Tau (τ)**:** $\tau = 1 - 4 \int_0^1 \int_0^1 C_{,1}(u, v)C_{,2}(u, v)dudv$.
Hoeffding's Dependence Index (Φ)**:** $\Phi = 90 \int_0^1 \int_0^1 (C(u, v) - uv)^2 dudv$.
Pearson's Coefficient of Mean Square Contingency (ϕ^2)**:** $\phi^2 = \int_0^1 \int_0^1 [c(u, v) - 1]^2 dudv$.

The Bernstein copula has zero coefficient of tail dependence. Nevertheless, Sanchetta and Satchell (2004) have indicated that the Bernstein copula could capture increasing dependence as one moved to the tails.

Financial Application: Determination of Value at Risk and Expected Shortfall:

Copulas have been applied to the assessment of the Value at Risk (VaR) and the Expected Shortfall (ES)risk-measures of a portfolio (Embrechts et al.(2002); Embrechts et al.(2003a, 2003b); Caillault and Guégan (2009)). For illustration purpose consider a portfolio of two assets. Let X and Y be their continuous returns over a common time horizon with distribution functions F_1 and F_2 , respectively. Let λ be the weight of X. Denote by Z the portfolio return, i.e.,

$$Z = \lambda X + (1-\lambda)Y,$$

with the corresponding distribution function $F_Z(z) = Pr[Z \leq z]$. The VaR_α is defined as the α-th quantile of the distribution $F_Z(z)$ and the $ES_\alpha = E[Z|Z \leq VaR_\alpha]$.

The distribution function of Z can be expressed in terms of the copula C associated with the joint distribution function of (X, Y) as

$$F_Z(z) = \int_0^1 \int_0^{F_1\left(\frac{z}{\lambda} - \frac{1-\lambda}{\lambda} F_2^{-1}(v)\right)} c(u,v)dudv,$$

where c is the copula-density. The Var_α is the solution of the equation $F_Z(z) = \alpha$ and

$$ES_\alpha = \frac{1}{\alpha}\int_0^1 \int_0^1 (\lambda F_1^{-1}(u) + (1-\lambda)F_2^{-1}(v)) I[\lambda F_1^{-1}(u) + (1-\lambda)F_2^{-1}(v) \leq Var_\alpha] c(u,v)dudv,$$

where $I[A]$ denotes the indicator function of the set A.

Let $v^* = F_1\left(\frac{z}{\lambda} - \frac{1-\lambda}{\lambda}F_2^{-1}(v)\right)$, then using the approximating empirical Bernstein copula:

$$F_B(z) = \sum_{i=1}^{m}\sum_{j=1}^{m} C_n\left(\frac{i}{m}, \frac{j}{m}\right)\left\{\int_0^1 \frac{j - mv}{v(1-v)} P_{m,i}(v^*) P_{m,j}(v)dv\right\}.$$

One possible numerical solution for the Value at Risk can be obtained as follows. First, simulate V_{ijk} from Uniform $(0, 1)$, $k = 1, \cdots, K$, for large K, and $i, j = 1, \cdots, m$. Secondly, solve numerically or graphically the following equation for z,

$$\alpha = \sum_{i=1}^{m}\sum_{j=1}^{m} C_n\left(\frac{i}{m}, \frac{j}{m}\right)\left\{\frac{1}{K}\sum_{k=1}^{K}\left[\frac{j - mV_{ijk}}{V_{ijk}(1-V_{ijk})} P_{m,i}(V_{ijk}^*) P_{m,j}(V_{ijk})\right]\right\},$$

where $V_{ijk}^* = F_1\left(\frac{z}{\lambda} - \frac{1-\lambda}{\lambda}F_2^{-1}(V_{ijk})\right)$. Note that solving this equation seems lengthy but it is straightforward and can be useful when simulation from a copula becomes difficult. Moreover, this approximation can be used for any continuous and increasing function of a portfolio return.

The ES_α can be approximated by

$$\begin{aligned} &ES_\alpha(B_m) \\ &\quad = \tfrac{1}{\alpha}\textstyle\sum_{i=1}^m \sum_{j=1}^m C_n\left(\frac{i}{m}, \frac{j}{m}\right)\left\{\frac{\lambda}{K}\sum_{k=1}^K F_1^{-1}(U_{ijk})\left[\frac{i-mU_{ijk}}{U_{ijk}(1-U_{ijk})} P_{m,i}(U_{ijk}) P_{m,j}(U_{ijk}^*)\right]\right\} \\ &\quad + \tfrac{1}{\alpha}\textstyle\sum_{i=1}^m \sum_{j=1}^m C_n\left(\frac{i}{m}, \frac{j}{m}\right)\left\{\frac{1-\lambda}{K}\sum_{k=1}^K F_2^{-1}(V_{ijk})\left[\frac{j-mV_{ijk}}{V_{ijk}(1-V_{ijk})} P_{m,i}(V_{ijk}^*) P_{m,j}(V_{ijk})\right]\right\}, \end{aligned}$$

where U_{ijk} and V_{ijk}, $k = 1, \cdots, K$, for large K, and $i, j = 1, \cdots, m$, are independent variables simulated from Uniform $(0, 1)$, $V_{ijk}^* = F_1\left(\frac{VaR_\alpha}{\lambda} - \frac{1-\lambda}{\lambda}F_2^{-1}(V_{ijk})\right)$ and $U_{ijk}^* = F_2\left(\frac{VaR_\alpha}{\lambda} - \frac{1-\lambda}{\lambda}F_1^{-1}(U_{ijk})\right)$.

Data Analysis: Co movement of International Stock Markets

The study of the interdependence between international stock markets is crucial to exploit diversification benefits and to have a clear picture of global economic and financial integration. For such purpose, data on six international stock market indexes: S&P-500 (USA), DOWJONES (USA), DAX (Germany), CAC40 (France), NIKKEI (Japan) and FTSE100 (UK) are considered. The data consist of 380 monthly closing values of the indexes over the period 31 January 1973 through 31 August 2004. Results on the sample versions of Kendall's Tau and Spearman's rank correlation and their empirical Bernstein copula approximations are presented in Table 1. As can be seen from the Table 1, the estimates of the corresponding measures of dependence are very close to each other.

Table 1: Monthly International Stock Market Dependence Measures

	S&P-500	DOWJONES	DAX	CAC40	NIKKEI	FTSE100
S&P-500		0.7957*	0.2584	0.3290	0.3696	0.4157
		0.7805**	0.2561	0.3250	0.3658	0.4116
DOWJONES	0.9396^{+}		0.2517	0.3398	0.3697	0.4071
	0.9295^{++}		0.2489	0.3366	0.3643	0.4029
DAX	0.3749	0.3630		0.2214	0.2318	0.1999
	0.3716	0.3599		0.2186	0.2295	0.1978
CAC40	0.4623	0.4782	0.3211		0.4424	0.3253
	0.4578	0.4739	0.3188		0.4375	0.3208
NIKKEI	0.5178	0.5130	0.3401	0.6036		0.3713
	0.5125	0.5080	0.3378	0.5977		0.3671
FTSE100	0.5746	0.5635	0.2931	0.4580	0.5214	
	0.5691	0.5581	0.2909	0.4535	0.5160	

Note:Top right values refer to Kendall's Tau (*) and Kendall's Tau using Bernstein Copula Approximation for m=300 (**). Bottom left values refer to Spearman's Rho (+) and Spearman's Rho using Bernstein Copula Approximation for m=300 (++).

To exploit the smooth approximation of the Bernstein copula we further looked at the data recorded as bimonthly by taking the closing values of the indexes for the months: January, March, May, July, September and November. The sample version and the empirical Bernstein copula approximates of the Spearman's rank correlations are given in Table 2. It is observed that the bimonthly smoothed approximate correlations based on the empirical Bernstein copula are closer to the monthly ones as compared to the sample version for the data under consideration.

Table 2: Bimonthly International Stock Market Dependence Measures.

	S&P-500	DOWJONES	DAX	CAC40	NIKKEI	FTSE100
S&P-500		$0.9342^{\sharp}$	0.3898	0.5053	0.5702	0.6321
DOWJONES	$0.8990^{\$}$		0.3747	0.5256	0.5344	0.6176
DAX	0.3568	0.3416		0.2668	0.3771	0.3955
CAC40	0.4718	0.4919	0.2341		0.6393	0.4380
NIKKEI	0.5363	0.5006	0.3442	0.6052		0.5120
FTSE100	0.5979	0.5836	0.3622	0.4048	0.4785	

Note:Top right values refer to Spearman's Rho (#) rank correlations. Bottom left values refer to Spearman's Rho using Bernstein Copula Approximation($) for m=500.

Acknowledgment

The research of Fentaw Abegaz is supported by a scholarship program from Addis Ababa University and Ministry of Education, Ethiopia. U.V. Naik-Nimbalkar is supported by a grant from the Council of Scientific and Industrial Research, India. The work was done while V. Mandrekar was visiting the University of Pune.

REFERENCES

Artzner, P., Delbaen, F., Eber,J.M., Heath D. (1999). Coherent measures of risk. *Math. Finance* 9(3), 203-228.

Caillault, C. and Guégan, D. (2009). Forecasting VaR and Expected Shortfall using Dynamical Systems: A Risk Management Strategy. *Frontiers in Finance and Economics,* 6, 26-50.

Deheuvels, P. (1979). La Fonction de Dépendence Empirique er ses Propriétés. Un Test Non paramétrique D'indépendence. *Acad. Roy. Belg. Bul. Cl. Sci.* (5)65, 274-292.

Deheuvels, P. (1981). A Nonparametric Test for Independence. *Publ Inst Statist Univ Paris*, 26, 29-50.

Durrleman, V., Nikeghbali, A. and Roncalli, T. (2000). Copulas Approximation and New Families. Groupe de Recherche Operationnells, Credit Lyonnais. Working Paper.

Embrechts, P., McNeil, A., Straumann, D. (2002). Correlation and dependence in risk management: properties and pitfalls. In M.A.H. Dempster (Ed.), *Risk Management: Value at Risk and Beyond*, 176-223. Cambridge University Press, Cambridge.

Embrechts, P., Hnig, A., and Juri , A. (2003a). Using Copula to Bound the Value at Risk for Functions of dependent risks. *Finance and Stochastics,* 7, 145-167.

Embrechts, P., Lindskog, F., and McNeil, A. (2003b). Modelling Dependence with Copulas and Applications to Risk Management. In S. Rachev (Ed.), *Handbook of Heavy Tailed Distributions in Finance.*, pp. 329-384,, Elsevier.

Feller, W. (1971). An Introduction to Probability Theory and Its Applications. Vol II, 2nd Edn., Wiley, New York.

Fermanian, J.-D., Radulovic D., and Wegkamp, M. H. (2004). Weak convergence of empirical copula processes. *Bernoulli* 10, 847-860.

Gzyl, H. and Palacios, J. L. (2003). On the Approximation Properties of Bernstein Polynomials via Probabilistic Tools. *Boletin de la Associatin Matemtica Venezolana,* X, 1, 5-13.

Hoeffding, W. (1994). Scale-Invariant Correlation Theory. N. I. Fisher and P. K. Sen (Eds.), *The Collected Works of Wassily Hoeffding.* 57-107, Springer Series in statistics.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association* 58 (301), 13-30.

Lojasiewicz S. (1988). *Introduction to the Theory of Real Functions.* Wiley, Chichester.

Li, X., Mikusiński P. and Taylor M. D. (1998). Strong Approximation Copulas. *J. of Mathematical Analysis and Applications,* 225, 608-623.

Nelsen (1999). *An Introduction to Copulas.* Lecture Notes in Statistics, 139, Springer Verlag , New York.

Olsen, E. T., Darsow W. F. and Nguyen B. (1996). Copulas and Markov Operators. In I. Rüschendorf, B. Schweizer and M. D. Taylor (Eds.), *"Proceedings of the Conference on Distributions with fixed Marginals and related topics".* IMS Lecture Notes and Monograph Series, 28, 244-259.

Sancetta, A. and Satchell, S. (2004). The Bernstein Copula and Its Applications to Modeling and Approximations of Multivariate distributions. *Econometric Theory,* 20, 535-562.

Sklar, A. (1959). Functions de répartition n dimensions et leurs marges, *Publ Inst Statist de Univ Paris,* 8, 229-231.