On the First Exit Time of a Nonnegative Markov Process Started at a Quasistationary Distribution

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Introduction

A recurrent stochastic process tends to visit "everywhere" as time progresses. Often there is interest in the time it takes the process to reach a taboo set. The probability that the process will not enter the taboo set within t time units tends to 0 as $t \to \infty$. Nonetheless, there may be interest in the conditional distribution of the process at time t, conditional on the process not having entered the taboo set by time t. If this distribution has a limit as $t \to \infty$, the limit is referred to as a quasi-stationary distribution.

In the context of first-exit times of Markov processes, quasi-stationary distributions come up naturally: an evaluation of the distribution of the state of a process upon its first entrance into a taboo set (if it took a long time to get there) can be obtained by conditioning on the state of the process before crossing (which approximately is the quasi-stationary distribution). These are of special interest in certain applications in the case of a nonnegative Markov process, where the first time that the process exceeds a fixed level A signals that some action is to be taken. The taboo set is (A, ∞), and the quasi-stationary distribution $\mathsf{Q}_A(x)$ is the distribution of the state of the process if a long time has passed and yet no crossover of A has occurred.

Various topics pertaining to quasi-stationary distributions are existence, calculation, simulation, etc. In the case of a Markov chain with a finite state space, a quasi-stationary distribution is a limit of high powers of a probability transition matrix, but in richer spaces the calculation of a quasi-stationary distribution often is not expressable analytically (see Tartakovsky, Pollak, and Polunchenko, 2011), and can only be approximated, perhaps by simulation. Even existence of a quasi-stationary distribution can be a vexing problem. For an extensive bibliography see Pollett (2008).

In this paper, we are interested in the dependence of certain characteristics of quasi-stationary distributions on the crossing threshold A when the process is nonnegative Markov. When an analytic expression of the quasi-stationary distribution is unavailable, this can be difficult. An example of such a characteristic appears in Pollak and Siegmund (1986), where it is shown, under certain conditions, that if a stationary distribution Q exists, then $\mathsf{Q}_A \to \mathsf{Q}$ as $A \to \infty$.

Here, we study a monotonicity property of the quasi-stationary distribution Q_A and apply it to the behavior of the expected time of the first exceedance of A by a Markov process started at

Q_A, as a function of A. Specifically, we provide conditions under which Q_A is nonincreasing and the corresponding stopping time $T_A^{\mathsf{Q}_A}$ is stochastically nondecreasing in A. While this is of considerable interest on its own merit, our interest stems from certain aspects in changepoint detection theory where it is of importance to establish monotonicity properties of the mean time to false alarm (as a function of the detection threshold) of detection schemes that start off at a random point that has a quasi-stationary distribution.

Results and discussion

Let $\{M_n\}_{n\geqslant 0}$ be an irreducible homogeneous Markov process taking values in $\mathscr{M} \subseteq [0,\infty)$ with transition probabilities $\rho(t,x) = \mathsf{P}(M_{n+1} \leqslant x | M_n = t)$. Define $T_A = \inf\{n \geqslant 0 : M_n > A\}$.

Assume that:

(C1) The quasi-stationary distribution $\mathsf{Q}_A(x) = \lim_{n\to\infty} \mathsf{P}(M_n \leqslant x | T_A > n)$ exists for all $A > A_0 \geqslant 0$ (for some $A_0 < \infty$) and satisfies $\mathsf{Q}_A(0) = 0$.

(C2) $\rho(s,x)$ is nonincreasing in s for all fixed $x \in \mathscr{M}$.

(C3) $\rho(ts,tx)$ is nondecreasing in t for all fixed $s,x \in \mathscr{M}$.

(C4) $\rho(s,x)/\rho(s,A)$ is nonincreasing in s for all fixed $x \in \mathscr{M}, x \leqslant A$.

(C5) $\rho(ts,tx)/\rho(ts,tA)$ is nondecreasing in t for all fixed $s,x, \in \mathscr{M}, x \leqslant A$.

Consider the case where M_0 has distribution Q_A and define

$$T_A^{\mathsf{Q}_A} = \inf\{n \geqslant 1 : M_n > A;\ M_0 \sim \mathsf{Q}_A\}.$$

Theorem. *Let the conditions* (C1)–(C5) *be satisfied. Then*
(i) M_0 *is stochastically nondecreasing in A; i.e.,* $\mathsf{Q}_{A_1}(x) \geqslant \mathsf{Q}_{A_2}(x)$ *for all x if* $A_1 < A_2$*;*
(ii) $\mathsf{Q}_{yA}(yx) \geqslant \mathsf{Q}_A(x)$ *for all $y \geqslant 1$ and all fixed* $x \in \mathscr{M}, x \leqslant A$*;*
(iii) $T_A^{\mathsf{Q}_A} \overset{\text{st}}{\preceq} T_{yA}^{\mathsf{Q}_{yA}}$ *for all $y \geqslant 1$, where* $\overset{\text{st}}{\preceq}$ *stands for "stochastically smaller than (or equal to)". In particular, it follows that* $\mathsf{E}[T_A^{\mathsf{Q}_A}] \leqslant \mathsf{E}[T_{yA}^{\mathsf{Q}_{yA}}]$ *for all $y \geqslant 1$.*

Although the conditions (C1)–(C5) are restrictive, they are satisfied in a number of interesting cases, some of which are provided below.

Suppose $\{M_n\}_{n\geqslant 0}$ obeys a recursion of the form

$$M_{n+1} = \varphi(M_n) \cdot \Lambda_{n+1}, \quad n = 0,1,\dots,$$

where

(D1) $\{\Lambda_i\}_{i\geqslant 1}$ are iid positive and continuous random variables;

(D2) the distribution function F of Λ_i satisfies

$$\frac{F(tx)}{F(tA)} \text{ increases in } t,\ t>0 \text{ for fixed } x \in \mathscr{M}, x \leqslant A;$$

(D3) $\varphi(t)$ is continuous, positive and nondecreasing in t;

(D4) $t/\varphi(t)$ is nondecreasing in t;

(D5) φ and F are such that $\mathsf{P}(\lim_{n\to\infty} M_n = 0) = 0$.

In this quite general example,

$$\rho(s, x) = F\left(\frac{x}{\varphi(s)}\right).$$

Under these conditions, Theorem III.10.1 of Harris (1963) can be applied to obtain existence of a quasi-stationary distribution. The conditions (D1)–(D5) are easily seen to imply the conditions (C1)–(C5).

Condition (D2) is equivalent to the log of the cdf of $\log(\Lambda_1)$ being concave. This is satisfied, for example, if $\log(\Lambda_1) = aY + b$ where a, b are real numbers and Y has a normal or an exponential distribution.

Many Markov processes fit this model. We now give several examples.

Example 1: EWMA (Exponentially weighted moving average) processes. Let

$$Y_{n+1} = \alpha Y_n + \xi_{n+1}, \quad n \geqslant 0,$$

where $0 \leqslant \alpha < 1$ and ξ_i, $i = 1, 2, \ldots$ are iid continuous random variables. Define $M_n = e^{Y_n}$, $\Lambda_n = e^{\xi_n}$. Here $\varphi(t) = t^\alpha$.

Example 2: Reflected random walk. Let

$$Y_0 = 0, \quad Y_{n+1} = (Y_n + Z_{n+1})^+, \quad n = 0, 1, \ldots,$$

where $\{Z_i\}$ are iid, $\mathsf{P}(Z_i < 0) > 0$. On the positive half plane the trajectory of the reflected random walk $\{Y_n\}_{n\geqslant 0}$ is identical to the trajectory of the Markov process $\{Y_n^*\}_{n\geqslant 0}$ given by the recursion

$$Y_0^* = 0, \quad Y_{n+1}^* = (Y_n^*)^+ + Z_{n+1}, \quad n = 0, 1, \ldots$$

Therefore, if $\log A > 0$ one may operate with Y_n^* instead of Y_n and all conclusions will be the same. Define $M_n = e^{Y_n^*}$ and $\Lambda_i = e^{Z_i}$, so that

$$M_{n+1} = \max(M_n, 1)\Lambda_{n+1}, \quad n \geqslant 0.$$

Here $\varphi(t) = \max(1, t)$. This process describes certain queuing systems as well as the Cusum change-point detection procedure. In the latter case, $Z_i = \log[f_1(X_i)/f_0(X_i)]$ is a log-likelihood ratio, where the observations $\{X_i\}_{i\geqslant 1}$ are iid with density f_0, and the goal is to detect an abrupt change from density f_0 to density f_1.

Example 3: Shiryaev-Roberts type Markov processes. Let $a > 0$ and $\varphi(t) = t + a$, so that $M_{n+1} = (M_n + a)\Lambda_{n+1}$. When $a = 1$ and $\Lambda_{n+1} = f_1(X_{n+1})/f_0(X_{n+1})$ is a likelihood ratio, where X_i, f_0 and f_1 are as in the previous example, $\{M_n\}_{n\geqslant 0}$ is a sequence of Shiryaev–Roberts statistics for detecting a change in distribution. The standard Shiryaev–Roberts procedure calls for setting $M_0 = 0$, specifying a threshold A and declaring at $T_A = \inf\{n \geqslant 1 : M_n > A\}$ that a change took place. A procedure $T_A^{\mathsf{Q}_A}$ that starts at a random point $M_0 \sim \mathsf{Q}_A$ has certain asymptotic optimality properties (cf. Pollak, 1985).

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ABSTRACT

Let $\{M_n\}_{n\geqslant 0}$ *be a nonnegative homogeneous Markov process and let*

$$\mathsf{Q}_A(x) = \lim_{n\to\infty} \mathsf{P}(M_n \leqslant x | M_0 \leqslant A, M_1 \leqslant A, \dots, M_n \leqslant A), \quad A > 0$$

be the corresponding quasi-stationary distribution. Suppose M_0 *has distribution* Q_A *and define* $T_A^{\mathsf{Q}_A} = \inf\{n \geqslant 1 : M_n > A\}$, *the first time when* M_n *exceeds* A. *We provide sufficient conditions for* $\mathsf{Q}_A(x)$ *to be nonincreasing in* A *(for fixed* x*) and for* $T_A^{\mathsf{Q}_A}$ *to be stochastically nondecreasing in* A.