Generalized Logistic Models and its orthant tail dependence

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1. The model

Let $\mathcal{L}(Z|W)$ denotes the conditional distribution of a random variable or vector Z given another random variable or vector W. For the vectors $\mathbf{X}_j = (X_{j,1}, \ldots, X_{j,d})$, $j = 1, ..., q$, and $\mathbf{S} = (S_1, \ldots, S_q)$, defined on the same probability space, we shall assume that:

(a) $\mathcal{L}((\mathbf{X}_1, ..., \mathbf{X}_q)|\mathbf{S}) = \prod_{j=1}^{q} \mathcal{L}(\mathbf{X}_j|\mathbf{S})$,

(b)$\mathcal{L}(\mathbf{X}_j|\mathbf{S}) = \mathcal{L}(\mathbf{X}_j|S_j)$,

(c) $P\left(\bigcap_{i=1}^{d} X_{ji} \le x_i | S_j\right) = C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j} S_j}, ..., e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j} S_j}\right)$, $x_j > 0$, $j = 1, ..., d$, where C_j's are max-stable copulas and $\{\beta_{ji}, j = 1, ..., q, i = 1, ..., d\}$ are non-negative constants such that $\sum_{j=1}^{q} \beta_{ji} = 1$, $i = 1, ..., d$,

(d) $E\left(e^{-tS_j}\right) = e^{-t^{\alpha_j}}$, $t \ge 0$, $j = 1, ..., q$, where α_j's are constants in $(0, 1]$

and

(e) $\mathcal{L}(\mathbf{S}) = \prod_{j=1}^{q} \mathcal{L}(S_j)$.

Thus every X_{ji} is a scale mixture with mixing variable $\beta_{ji} S_j^{\alpha_j}$ and $\mathbf{X}_j$, $j = 1, ..., q$, are conditionally independent given $\mathbf{S}$.
Scale mixtures have been studied and used in a variety of applications (Marshall and Olkin (1988, [7]), Joe and Hu (1996, [5]) and Fougères et al. (2009, [2]), Li (2009, [6])).

We shall consider here a componentwise maxima model from the $\mathbf{X}_j$'s. From this model we derive a new family of copulas and analize its orthant tail dependence by computing the multivariate tail dependence coefficients considered in Li (2009, [6])). Finally we apply the results to the particular case of C_j being the copula arising from the distribution of the variables in a M4 process (Smith and Weissman, 1996, [10]).

Proposition 1 *If the random vectors* $\mathbf{X}_j$, $j = 1, ..., q$, *and* $\mathbf{S}$ *satisfy the conditions (a)-(e) then* $\mathbf{Y} = (Y_1, \ldots, Y_d)$ *defined by* $Y_i = \bigvee_{j=1}^{q} X_{ji}$, $i = 1, ..., d$, *has multivariate extreme value distribution with* *unit Frechet margins and copula*

$$(1) \qquad C_{\mathbf{Y}}\left(u_1,...,u_d\right) = \exp\left\{-\sum_{j=1}^{q}\left(-\ln C_j\left(e^{-\left(-\beta_{j1}\ln u_1\right)^{1/\alpha_j}},...,e^{-\left(-\beta_{jd}\ln u_d\right)^{1/\alpha_j}}\right)\right)^{\alpha_j}\right\}.$$

Proof. To obtain $C_{\mathbf{Y}}$ we just apply the conditional independence of the $\mathbf{X}_j$'s followed by the max-stability of C_j's and the α_j-stability of each S_j, as follows:

$$P\left(\bigcap_{i=1}^{d}\{Y_i\le x_i\}\right) = \int P\left(\bigcap_{j=1}^{q}\bigcap_{i=1}^{d}\{X_{ji}\le x_i\}\,|\mathbf{S}=\mathbf{s}\right)d\mathbf{S}\left(s_1,...,s_q\right) =$$

$$\int\prod_{j=1}^{q}C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j}s_j},...,e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j}s_j}\right)d\mathbf{S}\left(s_1,...,s_q\right) =$$

$$\prod_{j=1}^{q}\exp\left\{-\left(-\ln C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j}},...,e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j}}\right)\right)^{\alpha_j}\right\}.$$

For each j and i, X_{ji} is a positive α_j-stable size mixture of a Fréchet distribution with location β_{ji}, scale $\beta_{ji}\alpha_j$ and shape parameter α_j and has itself Fréchet distribution with same location and the same right end point, but scale β_{ji} and shape parameter 1. Since $\sum_{j=1}^{q}\beta_{ji}=1$, $i=1,...,d$, each Y_i has unit Fréchet distribution. The max-stability of $C_{\mathbf{Y}}$ follows from its expression and the max-stability of the C_j's. ■

We now discuss some particular cases of (1) that has been explored.

(I) If $q=1$ then $\beta_{1i}=1$, $i=1,...d$, and

$$C_{\mathbf{Y}}\left(u_1,...,u_d\right) = \exp\left\{-\left(-\ln C_1\left(e^{-\left(-\ln u_1\right)^{1/\alpha_1}},...,e^{-\left(-\ln u_d\right)^{1/\alpha_1}}\right)\right)^{\alpha_1}\right\}$$

is a generalisation of the Archimedean copula (Joe, 1997, [4]), which for the particular case of the product copula $C_1=\Pi$ leads to the Gumbel-Hougaard or logistic copula. The dependence properties of the special case of $C_1(u_1,...,u_d) = \prod_{1\le s<t\le d}C_{\{s,t\}}(u_s^{p_s},u_t^{p_t})\prod_{i=1}^{d}u_i^{p_i\nu_i}$, where $C_{\{s,t\}}$, $1\le s<t\le d$, are bivariate copulas and $(d-1)p_i+p_i\nu_i=1$, $i=1,...,d$, were analysed in Joe and Hu (1996, [5]).

(II) If $C_j=\Pi$, $j=1,...,q$, then

$$\prod_{j=1}^{q}\exp\left\{-\left(-\ln C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j}},...,e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j}}\right)\right)^{\alpha_j}\right\} = \exp\left\{-\sum_{j=1}^{q}\left(\sum_{i=1}^{d}\left(\frac{x_i}{\beta_{ji}}\right)^{-1/\alpha_j}\right)^{\alpha_j}\right\},$$

which leads to an asymmetric logistic copula

$$(2) \qquad C_{\mathbf{Y}}\left(u_1,...,u_d\right) = \exp\left\{-\sum_{j=1}^{q}\left(\sum_{i=1}^{d}\left(-\beta_{ji}\ln u_i\right)^{-1/\alpha_j}\right)^{\alpha_j}\right\}.$$

In (2), if we take $\alpha_j = \alpha$, $j = 1, ..., q \leq +\infty$, we find an analogous mixture of extreme value distributions to those considered in Fougères et al. (2009, [2]) by departing just from a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ satisfying

$$\mathcal{L}(\mathbf{X}_j|\mathbf{S}) = \prod_{i=1}^{d} \mathcal{L}(X_i|\mathbf{S})$$

and

$$P(X_i \leq x|\mathbf{S}) = \exp\{-\left(\sum_{j=1}^{q} c_{ji} S_j\right)\left(1 + \gamma_i \frac{x - \mu_i}{\sigma_i}\right)^{-1/\gamma_i}\}, \quad i = 1, ..., d.$$

That this, in this different approach, conditionally on $\mathbf{S}$, the vector $\mathbf{X}$ has independent margins and each margin is a power mixture of an extreme value distribution with mixing variable $\sum_{j=1}^{q} c_{ji} S_j$, where the c_{ji} are non-negartive constants.

(III) Assume now, in (2), that each j corresponds to an element A of the set $\mathcal{S}$, the class of all nonempty subsets of $D = \{1, ..., d\}$. If $\beta_{Ai} = 0$ for each $i \notin A$ then the copula (2) becomes

(3) $$C_{\mathbf{Y}}(u_1, ..., u_d) = \exp\left\{-\sum_{A \subset \mathcal{S}} \left(\sum_{i \in A}^{d} (-\beta_{Ai} \ln u_i)^{-1/\alpha_A}\right)^{\alpha_A}\right\},$$

with $\sum_{A \subset \mathcal{S}} \beta_{Ai} = 1$, $i = 1, ..., d$. This is the asymmetric logistic model considered in Tawn (1990, [11]), by following a different probabilistic approach. More generally, by applying the same interpretation of the constants β_{ji} in (1), we obtain

(4) $$C_{\mathbf{Y}}(u_1, ..., u_d) = \exp\left\{\sum_{A \subset \mathcal{S}} \left(-\ln C_A\left(e^{-(-\beta_{Ai_1(A)} \ln u_1)^{1/\alpha_A}}, ..., e^{-(-\beta_{Ai_s(A)} \ln u_d)^{1/\alpha_A}}\right)\right)^{\alpha_A}\right\},$$

where C_A's are copulas with different dimensions and we denote by $(i_1(A), ..., i_s(A))$ the sub-vector of $(1, ..., d)$ corresponding to indices in A. In particular, if we begin with one copula $C_j = C$, $j = 1, ..., q$, then C_A, $A \subset \mathcal{S}$, are all the sub-copulas of C.

(IV) Finally, let us suppose that $\beta_{ji} = \beta_j$, $i = 1, ..., d$, in (1). Then

(5) $$C_{\mathbf{Y}}(u_1, ..., u_d) = \prod_{j=1}^{q} \exp\left\{-\left(\ln C_j\left(e^{-(-\ln u_1)^{1/\alpha_j}}, ..., e^{-(-\ln u_d)^{1/\alpha_j}}\right)\right)^{\alpha_j} \beta_j\right\},$$

with $\sum_{j=1}^{q} \beta_j = 1$, that is, $C_{\mathbf{Y}}$ is a geometric mean of mixtures of powers of multivariate extreme value distributions. The particular case of the weighted geometric mean $C_{\mathbf{Y}}(u_1, u_2) = (u_1 \wedge u_2)^{\beta_1}(u_1 u_2)^{1-\beta_1}$ is due to Cuadras and Augé (1981, [1]).

2. Orthant tail dependence

For a random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$ with continuos margins $F_1, ..., F_d$ and copula C, let the bivariate (upper) tail dependence parameters defined by

(6) $$\lambda_{\{s,t\}}^{(\mathbf{Y})} \equiv \lambda_{\{s,t\}}^{(C)} = \lim_{u \uparrow 1} P(F_s(Y_s) > u | F_t(Y_t) > u), \; 1 \leq s < t \leq d.$$

The tail dependence is a copula based measure and it holds

$$\lambda^{(C)}_{\{s,t\}} = 2 - \lim_{u \uparrow 1} \frac{\ln C_{\{s,t\}}(u,u)}{\ln u}, \tag{7}$$

where $C_{\{s,t\}}$ is the copula of the sub-vector (Y_s, Y_t) (Joe (1997, [4]), Nelsen (1999, [8])).

To characterise the relative strength of extremal dependence with respect to a particular subset of random variables of $\mathbf{Y}$ one can use conditional orthant tail probabilities of $\mathbf{Y}$ given that the components with indices in the subset J are extreme. The tail dependence of bivariate copulas can be extended as done in Schmid and Schmidt (2007) ([9]) and Li (2009) ([6]).

For $\emptyset \neq J \subset D = \{1, ..., d\}$, let

$$\lambda^{(\mathbf{Y})}_J \equiv \lambda^{(C)}_J = \lim_{u \uparrow 1} P\left(\bigcap_{j \notin J} \{F_j(Y_j) > u\} \mid \bigcap_{j \in J} \{F_j(Y_j) > u\} \right). \tag{8}$$

If for some $\emptyset \neq J \subset \{1, ..., d\}$ the parameter $\lambda^{(C)}_J$ exists and is positive then we say that $\mathbf{Y}$ is (upper) orthant tail dependent.

We have $\lambda^{(C)}_J = \frac{\lambda^{(C)}_{\{s\}}}{\lambda^{(C_J)}_{\{s\}}}$, if $\lambda^{(C_J)}_{\{s\}} \neq 0$ and the relation (7) between the tail dependence parameter and the bivariate copula can also be generalized by

$$\lambda^{(C)}_J = \lim_{u \uparrow 1} \frac{\sum\limits_{\emptyset \neq A \subset D} (-1)^{|A|-1} \ln C_A(\mathbf{u}_A)}{\sum\limits_{\emptyset \neq A \subset J} (-1)^{|A|-1} \ln C_A(\mathbf{u}_A)}, \tag{9}$$

where C_A denotes the sub-copula of C corresponding to margins with indices in A and and $\mathbf{u}_A$ the $|A|$-dimensional vector $(u, ..., u)$. By applying this relation and the max-stability of the copulas C_j, we get the following result.

Proposition 2 *For a copula C defined by (1), it holds*
(a)

$$\lambda^{(C)}_J = \frac{\sum\limits_{j=1}^{q} \sum\limits_{\emptyset \neq A \subset D} (-1)^{|A|-1} \left(-\ln C_{j,A}\left(e^{-\beta_{j1}^{1/\alpha_j}}, ..., e^{-\beta_{jd}^{1/\alpha_j}} \right)_A \right)^{\alpha_j}}{\sum\limits_{j=1}^{q} \sum\limits_{\emptyset \neq A \subset J} (-1)^{|A|-1} \left(-\ln C_{j,A}\left(e^{-\beta_{j1}^{1/\alpha_j}}, ..., e^{-\beta_{jd}^{1/\alpha_j}} \right)_A \right)^{\alpha_j}}, \tag{10}$$

where $C_{j,A}$ denotes the sub-copula of C_j corresponding to the margins with indices in A.

(b) If $C_j = \Pi$, for each $j = 1, ..., q$, then

$$\lambda^{(C)}_J = \frac{\sum\limits_{j=1}^{q} \sum\limits_{\emptyset \neq A \subset D} (-1)^{|A|-1} \left(\sum\limits_{i \in A} \beta_{ji}^{1/\alpha_j} \right)^{\alpha_j}}{\sum\limits_{j=1}^{q} \sum\limits_{\emptyset \neq A \subset J} (-1)^{|A|-1} \left(\sum\limits_{i \in A} \beta_{ji}^{1/\alpha_j} \right)^{\alpha_j}}. \tag{11}$$

The tail dependence result in (10) depends on the mixing variables through the parameters α_j, even for the case of $q = 1$, that is the global dependence added by the mixing variables doesn't vanish in extremes of maxima. This contrast with the result in Li (2009, [6]), where the scale mixture of MEV distributions $(RX_1, ..., RX_d)$ is considered with the mixing variable R satisfying $\frac{E(e^{-ctR})}{E(e^{-tR})} \to c^{-\alpha}$, as t tends to ∞, and $c \geq 1$, $\alpha > 0$. In this case the upper tail dependence parameters are exactly the same as the parameters of the MEV distribution without mixing.

We remark that, for $\beta_{ji} = \beta_j$, $i = 1, ..., d$, the numerator in (10) is, for each $A \subset D$,

$$\lambda_{\{s\}}^{(C_A)} = \sum_{j=1}^{q} \beta_j \lambda_{\{s\}}^{(C_{j,A})}$$

that is, the tail dependence parameter $\lambda_{\{s\}}^{(C_A)}$ is a linear convex combination of the corresponding tail dependence parameters for the sub-copulas $C_{j,A}$ of C_j, $j = 1, ..., q$.

The result in (11) leads to

$$\lambda_{\{s,t\}}^{(C)} = 2 - \sum_{j=1}^{q} \left(\beta_{js}^{1/\alpha_j} + \beta_{jt}^{1/\alpha_j}\right)^{\alpha_j},$$

extending the the known result

(12) $\qquad \lambda_{\{s,t\}}^{(C)} = 2 - 2^{\alpha},$

corresponding to $q = 1$ (Joe (1997, [4]), Nelsen (1999, [8])). The result in (10) enables to extend the equation (12) for other copulae C_1 than the product copula as

(13) $\qquad \lambda_{\{s,t\}}^{(C)} = 2 - (2 - \lambda_{\{s,t\}}^{(C_1)})^{\alpha}.$

3. Example

We will suppose that $C_j = C$, $j = 1, ..., d$, with

$$C(u_1, ..., u_d) = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} \left(\bigwedge_{i=1}^{d} u_i^{a_{lki}} \right), \quad u_j \in [0, 1], \ j = 1, ..., d,$$

where $\{a_{lkj}, l \geq 1, -\infty < k < \infty, 1 \leq j \leq d\}$, are nonnegative constants satisfying

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{lkj} = 1 \quad \text{for } j = 1, \ldots, d.$$

That copula arises from the common distribution of the variables of an M4 process (Smith and Weissman (1996, [10])).
Then the copula in (1) becomes

(14) $\qquad C_{\mathbf{Y}}(u_1, ..., u_d) = \exp\left\{ -\sum_{j=1}^{q} \left(\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \bigvee_{i=1}^{d} \left(-\beta_{ji} a_{lki}^{\alpha_j} \ln u_i\right)^{1/\alpha_j} \right)^{\alpha_j} \right\}.$

By applying the result in Proposition 2.1. (a), we obtain for the numerator in (10)

$$\lambda_{\{s\}}^{(C_{\mathbf{Y}})} = \sum_{j=1}^{q} \sum_{\emptyset \neq A \subset D} (-1)^{|A|-1} \left(\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \bigvee_{i \in A} \left(a_{lki} \beta_{ji}^{1/\alpha_j} \right) \right)^{\alpha_j} .$$

For the bivariate tail dependence it holds

$$\lambda_{\{s,t\}}^{(C_{\mathbf{Y}})} = 2 - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{j=1}^{q} \left(a_{lks} \beta_{js}^{1/\alpha_j} \vee a_{lkt} \beta_{jt}^{1/\alpha_j} \right)^{\alpha_j} ,$$

which, for the case $q = 1$ leads to the result $\lambda_{\{s,t\}}^{(C_{\mathbf{Y}})} = 2 - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} (a_{lks} \vee a_{lkt})$ in Heffernan et al. (2007, [3]).

Acknowledgments: This research was supported by the research unit "Centro de Matemática" of the University of Beira Interior and the research project PTDC/MAT/108575/2008 through the Foundation for Science and Technology (FCT) and co-financed by FEDER/COMPETE.

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