A general continuous auction system in presence of insiders

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Abstract

We analyze in a unified form how the presence of an insider makes the market be efficient when the insider knows the release time of the fundamental value of the asset. We also establish a general relationship between the problem of finding rational prices and the enlargement of filtrations problem. Moreover we consider the case when the time of the announcement is just a stopping time for all traders. In this latter case the market is not fully efficient, nevertheless there is an equilibrium where the sensitivity of prices is decreasing in time according with the probability that the announcement time is bigger than the current time. In others words, prices are becoming more and more stable when the announcement is coming.

1 The model

- We consider a financial market with a risky asset S and a bank account with interest rate $r = 0$. $V_t, t \geq 0$ indicates the fundamental value of S and $P_t, t \geq 0$, its market price.
- The trading period is $[0, \infty)$.
- At time τ (random) an announcement reveals the fundamental value of the S, V_τ, and the game is finished.

There are three kinds of traders:

- An insider who has information about V all the time,
- Noise traders who trade randomly
- Market makers, who set prices and clear the market.
- At time t, the information of the insider is given by $\mathcal{H}_t$, where $\mathcal{H}_t = \sigma(P_s, V_s, \tau \wedge s, 0 \leq s \leq t)$. We assume that V is a continuous $\mathbb{H}$-martingale, where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$. The insider tries to maximize his final wealth. His demand process is X and we assume that it is an $\mathbb{H}$-adapted process of the form

$$\mathrm{d}X_t = \theta_t \mathrm{d}t. \tag{1}$$

- Z is the demand process of the noise traders. We assume that Z is an $\mathbb{H}$-martingale, independent of V and $Z_0 = 0$.
- Finally, the market makers observe $Y = X + Z$, the total demand, and *clear* the market (so their demand process is $-Y$), and fix a *competitive* (rational) price, given by
$$P_t = \mathbb{E}(V_t | Y_s, 0 \leq s \leq t), 0 \leq t < \tau.$$

Market makers fix prices through a pricing rule, in terms of formulas,

$$P_t = H(t, \xi_t), 0 \leq t < \tau$$

with

$$\xi_t := \int_0^t \lambda(s) \mathrm{d}Y_s = \int_0^t \lambda(s)(\theta(s)\mathrm{d}s + \mathrm{d}Z_s)$$

where λ is a positive deterministic function and $H(t, \cdot)$ is strictly increasing for every t. We indicate a pricing rule by the pair (H, λ).

This model generalizes many models:

- If $V_t \equiv V$, $\tau \equiv 1$ and Z is a Brownian motion then we have Kyle-Back's model (1992).
- If $V_t \equiv V$, $\tau \equiv 1$ and Z is a Brownian motion with variance depending on time we have Aase, Bjuland and Øksendal's model (2007) .
- If $V_t \equiv V$ $\tau \equiv 1$ and Z is a Lévy process we have CFDO's model (2010).
- If $V_t \equiv V$ $\tau \equiv 1$ and $\mathrm{d}Z_t = a(Y_t)\mathrm{d}B_t$ where B is a Brownian motion, then we have the Campi, Danilova and Cetin's model (2010).
- If $V_t = \mathbf{1}_{\{\bar{\tau} > 1\}}$, $\bar{\tau}$ is an $\mathbb{H}$-stopping time, $\tau = \bar{\tau} \wedge 1$ and τ is known by the insider, this is the Campi and Cetin's model (2007).
- If V and Z are independent Brownian motions and τ an independent stopping time with exponential distribution, then we have Caldentey and Stachetti's model (2010)....

The wealth process

Consider first a discrete case where trades are made at times $i = 1, 2, \ldots N$, and where N is random. If at time $i - 1$, there is an order of buying $X_i - X_{i-1}$ shares, its cost will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total change is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and just after time N, there is the extra income: $X_N V_N$ (we assume that $X_0 = 0$).

So, the total wealth generated is

$$\begin{aligned} W_{N^+} &= -\sum_{i=1}^{N} P_i (X_i - X_{i-1}) + X_N V_N \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N} (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N \end{aligned}$$

Analogously, in the continuous model,

$$\begin{aligned} W_{\tau+} &= X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau X_{t-} \mathrm{d}V_t + \int_0^\tau V_{t-} \mathrm{d}X_t + [V, X]_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-}) \, \mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau \end{aligned}$$

2 The equilibrium

Definition 1. A demand process X is said to be optimal given a pricing rule (H, λ) if it maximizes $E(W_{\tau+})$.

Definition 2. The triple (H, λ, X) is an equilibrium, if the price process $P_t = H(t, \xi_t)$ is competitive, given X, and the strategy X is optimal, given (H, λ).

The wealth at time τ_+ is given by

$$W_{\tau+} := \int_0^\tau (V_t - P_t)\theta_t \mathrm{d}t + \int_0^\tau X_t \mathrm{d}V_t,$$

where $P_t = H(t, \xi_t)$, with $\xi_t := \int_0^t \lambda(s)(\theta(s)\mathrm{d}s + \mathrm{d}Z_s)$, and where Z is an $\mathbb{H}$-martingale.

Optimization

Then we want to maximize

$$J(\theta) := E\left(W_{\tau+}\right) = E\left(\int_0^\tau (V_t - H(t, \xi_t))\theta_t \mathrm{d}t\right),$$

then, if θ is optimal, for all β, such that $\theta + \varepsilon\beta$ is another insider's strategy, with $\varepsilon > 0$ small enough, we will have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(\theta+\varepsilon\beta) \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E\left(\int_0^\tau (V_t - H(t, \int_0^t \lambda_s((\theta_s+\varepsilon\beta_s)\mathrm{d}s + \mathrm{d}Z_s)))\,(\theta_t+\varepsilon\beta_t)\,\mathrm{d}t\right) \\
&= E\left(\int_0^\tau (V_t - H(t,\xi_t))\beta_t \mathrm{d}t\right) + E\left(\int_0^\tau -\partial_2 H(t,\xi_t)\theta_t \left(\int_0^t \lambda_s\beta_s \mathrm{d}s\right)\mathrm{d}t\right) \\
&= E\left(\left(\int_0^\tau (V_t - H(t,\xi_t)) - \lambda_t \int_t^\tau \partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\right)\beta_t \mathrm{d}t\right) \\
&= E\left(\left(\int_0^\infty \mathbf{1}_{[0,\tau]}(t)(V_t - H(t,\xi_t)) - \lambda_t \int_t^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\right)\beta_t \mathrm{d}t\right)
\end{aligned}$$

Then, since we can take $\beta_t = \mathbf{1}_{[u,u+h]}(t)\ \alpha_u$, with α_u H_u-measurable,

$$\begin{aligned}
&E\left(\alpha_u \int_u^{u+h} \left(\mathbf{1}_{[0,\tau]}(t)\,(V_t - H(t,\xi_t)\right)\right. \\
&\left.-\lambda_t \int_t^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\Big)\,\mathrm{d}t\right) = 0,
\end{aligned}$$

therefore

$$\begin{aligned}
&\int_0^u \big(E(\mathbf{1}_{[0,\tau]}(t)V_t|\mathcal{H}_t) - E(\mathbf{1}_{[0,\tau]}(t)H(t,\xi_t)|\mathcal{H}_t) \\
&- \lambda_t \int_t^\infty E\left(\mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_t\right)\mathrm{d}s\Big)\,\mathrm{d}t, \qquad u \geq 0
\end{aligned}$$

is an $\mathbb{H}$-martingale, and consequently

$$\begin{aligned}
&E(\mathbf{1}_{[0,\tau]}(t)V_t|\mathcal{H}_t) - E(\mathbf{1}_{[0,\tau]}(t)H(t,\xi_t)|\mathcal{H}_t) \\
&-\lambda_t \int_t^\infty E\left(\mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_t\right)\mathrm{d}s = 0, \quad t \geq 0.
\end{aligned}$$

Now, since τ is an $\mathbb{H}$-stopping time, and $\mathbb{F}^{P,V} \subseteq \mathbb{H}$, we can write, in the set $\{t < \tau\}$,

$$V_t - H(t,\xi_t) - \lambda_t E\left(\int_t^\tau \partial_2 H(s,\xi_s)\theta_s \mathrm{d}s|\mathcal{H}_t\right) = 0,$$

Moreover if τ is known at 0,

$$V_t - H(t,\xi_t) - \lambda_t \int_t^\tau E\left(\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_t\right)\mathrm{d}s = 0, \quad 0 \leq t \leq \tau.$$

and, in particular,

$$\lim_{t\to\tau} H(t,\xi_t) = V_\tau,$$

This is the case in Campi and Çetin (2007), where they take $V_t = \mathbf{1}_{\{\bar\tau>1\}}$, $\bar\tau$ is an $\mathbb{H}$-stopping time and $\tau = \bar\tau \wedge 1$ and τ is known by the insider, that is $\tau \in \mathcal{H}_0$. Then they obtain

$$\mathbf{1}_{\{\bar\tau>1\}} - H(\bar\tau \wedge 1, \xi_{\bar\tau\wedge 1}) = 0,$$

Also they assume that $\bar{\tau}$ is the first passage time of a standard Brownian motion that is independent of Z.

In Caldentey and Stacchetti (2010) authors assume that V is an arithmetic Brownian motion and τ follows an exponential distribution independent of V and P then, on the set $t < \tau$

$$V_t - H(t, \xi_t) - \lambda_t \int_t^\infty e^{-\mu(s-t)} E\left(\partial_2 H(s, \xi_s)\theta_s | \mathcal{H}_t\right) \mathrm{d}s = 0, a.e.$$

Price pressure

From the equation

$$V_t - H(t, \xi_t) - \lambda_t \int_t^\tau E\left(\partial_2 H(s, \xi_s)\theta_s | \mathcal{H}_t\right) \mathrm{d}s = 0, \quad 0 \le t \le \tau.$$

and taking into account that Z is an $\mathbb{H}$-martingale,

$$E\left(\int_t^\tau \partial_2 H(s, \xi_s)\theta_s \mathrm{d}s | \mathcal{H}_t\right) = E\left(\int_t^\tau \frac{1}{\lambda_s} \partial_2 H(s, \xi_s) \mathrm{d}\xi_s | \mathcal{H}_t\right),$$

and we can conclude that $\lambda(t) = \lambda_0$, and that $H(t, y)$ satisfies the equation (assuming, for simplicity, Z continuous)

$$\partial_1 H(s, y) + \frac{1}{2} \partial_{22} H(s, y) \lambda_s^2 \sigma_s^2 = 0,$$

and where $\sigma^2 := \frac{\mathrm{d}[Z,Z]_s}{\mathrm{d}s}$.

Competitive prices

Since

$$\partial_1 H(s, y) + \frac{1}{2} \partial_{22} H(s, y) \lambda_0^2 \sigma_s^2 = 0$$

we have that

$$\mathrm{d}P_t = \mathrm{d}H(s, \lambda_0 Y_t) = \partial_2 H(s, \lambda_0 Y_t) \lambda_0 \mathrm{d}Y_t,$$

in such a way that prices are competitive ($\mathbb{F}^Y$-martingale) if and only if Y is an $\mathbb{F}^Y$-martingale.

3 Enlargement of filtrations

The aggregate demand process Y is given by

$$Y_t = Z_t + \int_0^t \theta_s \mathrm{d}s, \quad t \ge 0$$

where Z is an $\mathbb{H}$-martingale and θ is $\mathbb{H}$-adapted. So,the r.h.s is the Doob-Meyer decomposition of the $\mathbb{F}^Y$-martingale w.r.t to the filtration $\mathbb{H} \supset \mathbb{F}^Y$.

Example 1. (Back 92) Assume that Z is a Brownian motion with variance σ^2 and $V \equiv V_1$. If the strategy is optimal $V_1 = H(1, Y_1)$, and if V_1 has continuous cdf we can assume that $Y_1 \equiv$N$(0, \sigma^2)$ by choosing $H(1, \cdot)$ conveniently. We assume that V_1 (and consequently Y_1) is independent of Z.

On the other hand, by the results of enlargement of filtrations we know that

$$Y_t = Z_t + \int_0^t \frac{Y_1 - Y_s}{1-s}\mathrm{d}s, \quad 0 \leq t \leq 1$$

is a Brownian motion with variance σ^2. So there is an equilibrium with the strategy

$$\theta_t = \frac{Y_1 - Y_t}{1-t}.$$

Example 2. (Aase *et. al* (2007))

$$Z_t = \int_0^t \sigma_s dW_s$$

where σ is deterministic and $V \equiv Y_1$ is a N$(0, \int_0^1 \sigma_s^2 \mathrm{d}s)$ independent of Z. Then

$$Y_t = Z_t + \int_0^t \frac{g'(s)(Y_s - Y_1)}{g(s)}\mathrm{d}s,$$

with $g(t) = \int_t^1 \sigma_s^2 \mathrm{d}s$, has the same law as Z. We have a similar result if σ is random.

Example 3. (Campi, Cetin , Danilova 2009) In fact if $dZ_t = \sigma(Y_t)dW_t$ and $V \equiv V_1$, the final value of $V_t = \int_0^t \sigma(V_s)dB_s$, and independent of Z, then

$$dY_t = \sigma(Y_t)dW_t + \sigma^2(Y_t)\frac{\partial_y G(1-t, Y_t, V_1)}{G(1-t, Y_t, V_1)}dt$$

where $G(t, y, z)$ is the transition density of $V_.$, is a martingale.

Example 4. (Campi and Cetin (2007)) If we want the aggregate process Y to be a Brownian motion that reaches the value -1 for the first time at time τ, and Z is also a Brownian motion then

$$Y_t = Z_t + \int_0^t \left(\frac{1}{1+Y_s} - \frac{1+Y_s}{\tau - s}\right)\mathbf{1}_{[0,\tau]}(s)\mathrm{d}s,$$

so, in this case $V \equiv \tau$.

Example 5. (Back and Pedersen (1998), Wu(1999), Danilova (2008)) The insider receives a continuous signal

$$V_t = V_0 + \int_0^t \sigma_s dW_s,$$

where V_0 is a zero mean normal r.v., W is a Brownian motion, both independent of Z, also a Brownian motion, σ is deterministic. It is assumed that $var(V_1) = var(V_0)) + \int_0^t \sigma_s^2 ds = 1$, then

$$Y_t = Z_t + \int_0^t \frac{V_t - Y_t}{var(V_t) - t}dt,$$

is a Brownian motion.

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