# The distribution of the product of beta determinants

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The main focus of this paper is to derive exact expressions for the distributions of the product of determinants of existing and new noncentral bimatrix variate beta distributions with bounded domain. This is done by first deriving the product moment of the corresponding noncentral bimatrix variate beta distribution followed by Mellin and inverse Mellin transforms.

A large variety of hypothesis tests in multivariate analysis make use of the likelihood ratio method to derive appropriate test criteria. Several of the test statistics used are functions of the determinant or product of determinants of matrix or bimatrix beta variates respectively (see Kshirsagar, 1972; Anderson, 1984). The best known test statistic in multivariate analysis is the Wilks' statistic (Wilks, 1932) defined as

$$\Lambda \equiv \left| \frac{\boldsymbol{S}}{\boldsymbol{S}+\boldsymbol{B}} \right| = |\boldsymbol{U}|$$

with $\boldsymbol{S}$ and $\boldsymbol{B}$ two independent $(p \times p)$ Wishart matrices, i.e. $\boldsymbol{S} \sim W_p\left(n, \boldsymbol{I}_p\right)$ and $\boldsymbol{B} \sim W_p\left(m, \boldsymbol{I}_p; \boldsymbol{\Theta}\right)$, $n, m \geq p$ and $\boldsymbol{\Theta}$ the noncentrality parameter. Note that $\boldsymbol{U} = (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}}$ has the *noncentral matrix variate beta type I distribution.* The distribution under the nonnull hypothesis is of importance when calculating the power of the test and Bekker, Roux and Arashi (2011) gave an exact expression for the nonnull distribution of the Wilks' statistic.

Firstly, we discuss the product of determinants of matrix variates having the *noncentral bimatrix variate beta type IV distribution.* Let $\boldsymbol{S}_i \sim W_p\left(n_i, \boldsymbol{I}_p\right)$, $i = 1, 2$, and $\boldsymbol{B} \sim W_p\left(m, \boldsymbol{I}_p; \boldsymbol{\Theta}\right)$ independent, then $\mathbb{X} = (\boldsymbol{X}_1 : \boldsymbol{X}_2)'$, where $\boldsymbol{X}_i = (\boldsymbol{S}_i+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_i+\boldsymbol{B})^{-\frac{1}{2}}$, $i = 1, 2$, has the *noncentral bimatrix variate beta type IV distribution.* In this presentation we will derive the exact expression for the density function of

$$(1) \qquad \Lambda_1 \equiv \left| \frac{\boldsymbol{S}_1}{\boldsymbol{S}_1+\boldsymbol{B}} \right| \left| \frac{\boldsymbol{S}_2}{\boldsymbol{S}_2+\boldsymbol{B}} \right| = |\boldsymbol{X}_1 \boldsymbol{X}_2| .$$

Note that if $\boldsymbol{B} \sim W_p\left(m, \boldsymbol{I}_p\right)$ then $\mathbb{X}$ has the central bimatrix variate beta type IV distribution, also known as the bimatrix variate generalised beta distribution (see Diaz-Garcia and Gutiérrez-Jáimez, 2010; Gupta and Nagar, 2009). For the latter case Bekker, Roux, Ehlers and Arashi (2011) derived an exact expression for the density function of $\Lambda_1$.

Let $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{I}_p)$, $i = 1, 2$, and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$ independent, and let $\boldsymbol{U}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}}$, $i = 1, 2$, then the distribution of $\mathbb{U} = (\boldsymbol{U}_1 : \boldsymbol{U}_2)'$ is known as the *noncentral bimatrix variate beta type I distribution.* The statistic

$$(2) \quad \Lambda_2 \equiv \left| \frac{\boldsymbol{S}_1}{\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}} \right|^{\frac{1}{2}n_1} \left| \frac{\boldsymbol{S}_2}{\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}} \right|^{\frac{1}{2}n_2} = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$$

arises when testing whether two normal populations are identical (Anderson, 1984). Building constant factors into the covariance matrices of the Wishart matrix variates leads to a generalised statistic and the definition of the *noncentral bimatrix variate beta type V distribution.* In this presentation the density function for this proposed distribution is derived as well as the density function for $\Lambda_3$, the generalised statistic resulting from this.

The density functions of $\Lambda_i$, $i = 1, 2, 3$ are complemented with graphical representations in the bivariate as well as the bimatrix case. Since exact expressions for the density functions of these statistics are now available, it is also possible to derive exact confidence intervals. These distributions add value to multivariate statistical analysis with specific reference to factors of Wilks' statistics and the product of generalised statistics.

## REFERENCES


1. Anderson, T.W. (1984). *An introduction to multivariate statistical analysis*, 2nd ed., John Wiley & Sons, New York.
2. Bekker, A. Roux, J.J.J. & Arashi, M. (2011). Exact nonnull distribution of Wilks's Statistic: ratio and product of independent components. *Journal of Multivariate Analysis*, 102, 619-628. (http://dx.doi.org/10.1016/j.jmva.2010.11.005)
3. Bekker, A. Roux, J.J.J., Ehlers, R. & Arashi, M. (2011). The bimatrix variate beta type IV distribution: relation to Wilks's statistic and bimatrix variate Kummer-beta type IV distribution. Accepted for publication *Communications in Statistics-Theory and Methods.* (Status: in production)
4. Díaz-García, J.A. and Gutiérrez-Jáimez, R. (2010). Bimatrix variate generalised beta distributions, *South African Statistical Journal*, 44, 193-208.
5. Gupta, A.K. and Nagar, D.K. (2009). Matrix variate generalization of a bivariate beta type I distribution. *Journal of Statistics and Management Systems*, 12 (5), 873-885.
6. Kshirsagar, A.M. (1972). *Multivariate Analysis*, Marcel Dekker, New York.
7. Wilks, S.S. (1932). Certain generalizations in the analysis of variance, *Biometrika*, 24, 471-494.