The theoretical development of a new multivariate beta distribution

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Practitioners in the quality control environment require methods to monitor a process from the start of the production, whether or not prior (historical or past) information is available for estimating the parameters. Quesenberry (1991) presented Q-charts assuming that the observations from each sample are independent and identically distributed normal random variables. However, there are cases where the assumption of normality is not valid. Human and Chakraborti (2010) proposed a Q-chart design for monitoring the process mean when the measurements are from an *exponential distribution* and the parameter of the distribution is unknown.

Consider successive independent samples of size n (assuming that these values are independent and identically distributed) collected from an $EXP(\theta)$ distribution where the parameter θ denotes the unknown process mean. At some point in time the process parameter will change from θ to $\lambda\theta$ where $\lambda > 0$; therefore if $\lambda = 1$ the process remains in control. Because θ is unknown the first sample is used to obtain an initial estimate of θ, assuming that the process started in control. This initial estimate is continuously updated using the new incoming samples as they are collected, as long as the value of θ does not change. The two-sample test statistic for testing the hypothesis at time r (the two independent samples are from exponential distributions with the same unknown parameter) is based on the ratio of the sample means.

To gain more insight into the performance of this control chart, one needs to consider the run-length distribution of the proposed chart. The run-length of a control chart is the number of samples collected until the shift is detected; this can also be viewed as the waiting time until a signal is observed following a shift or change in the process parameter. To develop exact expressions for the probabilities of the run-lengths the joint distribution of the charting statistics is needed. This results in a new multivariate distribution for this exponential case.

The charting statistic immediately after the change in the parameter θ is a function of a random variable denoted by U_0. U_1 denotes the next random variable, one time period after U_0, continuing the same way, U_p denotes the random variable p time periods after U_0. The form of the construction of these random variables with their dependence structure, that originated from the above-mentioned problem identified in Statistical Process Control (SPC), will be presented. The joint distribution of the random variables $U_0, U_1, \ldots, U_p$ is derived and can be regarded as a generalized multivariate beta distribution. This distribution depends on λ (which indicates the size of the shift), a (related to when the shift took place) and v (that depends on the sample size at each point in time). The presentation will also focus on the univariate and bivariate distributions for illustrative purposes, with emphasis on the correlation structure of the random variables.

This particular distribution is developed within the Statistical Process Control paradigm. The genesis of this new contribution to the field of multivariate distributions is presented. We will also address other related, interesting problems, with potential practical applications, that originated from this discussion.

REFERENCES

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