Inference for parameter instability in time series via trending regression

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Introduction

The talk concerns segmentation procedure for mean-nonstationary time series. Nonstationarities can arise in various ways our focus is on nonstationarity in the trend.
The considered problem is formulated in the framework of detecting changes in trending regression models in which the regressors are generated by suitably smooth functions and the error terms can be both independent or dependent. The proposed test procedures are related to likelihood ratio type statistics and statistics based on partial sums of weighted residuals.
Limit behavior of the proposed test procedures both under no change and at least one change is presented. The main theoretical results among others establish the extreme value distribution of these statistics. This provides a simple approximation for the needed critical values. However it appears that the convergence to the limit is rather slow therefore a suitable version of the circular bootstrap is proposed and then applied.

Test statistics

We study the regression model

$$\begin{aligned} Y_i \quad &= \boldsymbol{x}_i^T\boldsymbol{\beta} + e_i, \qquad && i = 1\ldots,k^*,\\ &= \boldsymbol{x}_i^T\boldsymbol{\beta} + \boldsymbol{x}_i^T\boldsymbol{\delta} + e_i, && i = k^*+1\ldots,n, \end{aligned}$$

where $e_1,\ldots,e_n$ are random errors with zero mean, $\boldsymbol{\beta},\boldsymbol{\delta} \neq \mathbf{0}$ are p-dimensional parameters, k^* is a change point (structural break) and $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n$ are p-dimensional design points (random or nonrandom).
The testing problem is formulated as follows:

$$H_0: \quad k^* = n \qquad versus \qquad H_1: \quad k^* < n,$$

i.e. no change versus there is a change at an unknown point $k^* < n$.

This is one of the basic formulation for testing no change versus there is a change in linear regression models. Of course, there is also the problem to estimate the location of the change point k^*. There are many papers and even books and survey papers on this problems, e.g., Csörgő and Horváth (1997), Perron (2008). There is typically assumed that the design points $\boldsymbol{x}_i$, $i = 1, \ldots, n$ have neither deterministic nor stochastic trends, mostly formulated as an assumption of either stationarity of $\{\boldsymbol{x}_j\}_j$ (random design) or for large n

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \boldsymbol{x}_i \boldsymbol{x}_i^T \approx t\boldsymbol{C}, \quad t \in (0,1)$$

for some positive definite matrix $\boldsymbol{C}$ (fixed design) .
Here we focus on situations where $\boldsymbol{x}_i$ have a trend:

$$\boldsymbol{x}_i = \boldsymbol{h}(i/n) = (h_1(i/n), \ldots, h_p(i/n))^T, \quad i = 1, \ldots, n,$$

where h_j, $j = 1, \ldots, p$, are smooth nonconstant functions. The same test statistics as in case of no trend in regression can be used, however their limit behavior is different. We will show it on some particular test procedures.
We will deal with two maxlikelihood type test statistics:

$$(1) \quad T_n(\eta) = \max_{\eta n \le k < n(1-\eta)} T_n = \max_{p \le k < n-p} \Big\{ \boldsymbol{S}_k^T \boldsymbol{C}_k^{-1} \boldsymbol{C}_n (\boldsymbol{C}_k^0)^{-1} \boldsymbol{S}_k \frac{1}{\widehat{\tau}_n^2} \Big\},$$

$$(2) \quad T_n = \max_{p \le k < n-p} \Big\{ \boldsymbol{S}_k^T \boldsymbol{C}_k^{-1} \boldsymbol{C}_n (\boldsymbol{C}_k^0)^{-1} \boldsymbol{S}_k \frac{1}{\widehat{\tau}_n^2} \Big\},$$

where

$$\boldsymbol{S}_k = \sum_{i=1}^{k} \boldsymbol{h}(i/n)(Y_i - \boldsymbol{h}^T(i/n)\boldsymbol{\beta}_n),$$

$$\boldsymbol{C}_k = \sum_{i=1}^{k} \boldsymbol{h}(i/n)\boldsymbol{h}^T(i/n), \quad \boldsymbol{C}_k^0 = \boldsymbol{C}_n - \boldsymbol{C}_k$$

and $\widehat{\tau}_n^2$ is suitable standardization quantity. This is the form more convenient for calculations. Alternatively they can be expressed as

$$(3) \quad T_n(\eta) = \max_{\eta n \le k < n(1-\eta)} \Big\{ \Big(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0\Big)^T \widehat{\boldsymbol{\Sigma}}_k^{-1} \Big(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0\Big) \Big\},$$

$$(4) \quad T_n = \max_{p < k < n-p)} \Big\{ \Big(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0\Big)^T \widehat{\boldsymbol{\Sigma}}_k^{-1} \Big(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0\Big) \Big\},$$

where $0 \le \eta < 1/2$, $\widehat{\boldsymbol{\beta}}_k$ and $\widehat{\boldsymbol{\beta}}_k^0$ are least squares estimators based on $Y_1, \ldots, Y_k$ and $Y_{k+1}, \ldots, Y_n$, respectively, and $\widehat{\boldsymbol{\Sigma}}_k^{-1}$ is the inverse matrix to a suitable chosen estimator of the variance matrix of $\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0$. Here the statistics T_n and $T_n(\eta)$ are expressed in terms of differences of the least squares estimator based on the first k and last $n-k$ observations statistics, which give the message that they can be sensitive w.r.t. changes in $\boldsymbol{\beta}$.

The null hypothesis is rejected for large values of either T_n or $T_n(\eta)$. Approximations of the critical values can be find in either of the following way:
(i) using limit distribution of T_n and/or $T_n(\eta)$ under H_0,
(ii) by a proper version of resampling methods (bootstrap).
Theorems 1 and 2 below provide the desired result for (i), while suitable version of the bootstrap is shortly discussed later.

Note that the change point k^* can be estimated as a the maximizer of

$$\left\{(\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0)^T \widehat{\boldsymbol{\Sigma}}_k^{-1} (\widehat{\boldsymbol{\beta}}_k - \widehat{\boldsymbol{\beta}}_k^0)\right\}$$

w.r.t. k.

Asymptotic results

Here we assume:

- (A.1) The sequence $(e_i \colon i \geq 1)$ satisfy $E[e_i] = 0, \qquad E[e_i^2] = \sigma^2 > 0$.
- (A.2) There are independent standard Brownian motions $(W_{1,n}(t) \colon t \geq 0)$ and $(W_{2,n}(t) \colon t \geq 0)$ such that

  $$(5) \qquad \max_{1 \leq k \leq n/2} \frac{1}{k^{1/\nu}} \left| \sum_{i=1}^{k} e_i - \tau W_{1,n}(k) \right| = \mathcal{O}_P(1) \qquad (n \to \infty)$$

  and

  $$(6) \qquad \max_{n/2 < k < n} \frac{1}{(n-k)^{1/\nu}} \left| \sum_{i=k+1}^{n} e_i - \tau W_{2,n}(n-k) \right| = \mathcal{O}_P(1) \qquad (n \to \infty)$$

  with some $\nu > 2$ and $\tau > 0$.
- (A.3) The components of $\boldsymbol{h}(.)$ are continuous on $[0,1]$. The matrices $\int_0^t \boldsymbol{h}(x)\boldsymbol{h}^T(x)dx$ and $\boldsymbol{C}(t) = \int_t^1 \boldsymbol{h}(x)\boldsymbol{h}^T(x)dx$ are regular for all $t \in (t^0, 1-t^0)$ for all $t_0 \in (0, 1/2)$.
- (A.4) There are p linearly independent p-dimensional vectors $\boldsymbol{a}_{01}, \ldots, \boldsymbol{a}_{0p}$ and nonnegative $0 \leq \gamma_{01} < \ldots < \gamma_{0p}$ such that

  $$(7) \qquad \limsup_{t \to 0_+} \frac{1}{t^{\gamma_{0p}+1}} \left| \boldsymbol{h}(t) - \sum_{\ell=1}^{p} \boldsymbol{a}_{0\ell} t^{\gamma_{0\ell}} \right| < \infty.$$

- (A.5) There are p linearly independent p-dimensional vectors $\boldsymbol{a}_{11}, \ldots, \boldsymbol{a}_{1p}$ and nonnegative $0 \leq \gamma_{11} < \ldots < \gamma_{1p}$ such that

  $$(8) \qquad \limsup_{t \to 1_-} \frac{1}{t^{\gamma_{1p}+1}} \left| \boldsymbol{h}(t) - \sum_{\ell=1}^{p} \boldsymbol{a}_{1\ell} t^{\gamma_{1\ell}} \right| < \infty.$$

We shortly discuss the assumptions. Assumptions (A.1), (A.2) concerns error terms. They cover both independent observations (in the situation $\tau^2 = \sigma^2$) as well as dependent ones covering a quite spectrum of various time series. Assumptions (A.3)-(A.4) on design points cover a number of useful situations. Here are two important cases:

Polynomial regression $\boldsymbol{h}(t) = (t^{\gamma_1}, \ldots, t^{\gamma_p})^T$, $t \in [0,1]$, $\quad 0 \leq \gamma_1 < \ldots < \gamma_p$

Harmonic regression $\boldsymbol{h}(t) = (\cos(2\pi t\omega_1), \sin(2\pi t\omega_1), \ldots, \cos(2\pi t\omega_p), \cos(2\pi t\omega_p))^T$, $t \in [0,1]$, $\omega_1, \ldots, \omega_p$ known.

Next we formulate main results for both test statistics under the null hypothesis H_0.

Theorem 1 *Let assumptions (A.1)- (A.3) be satisfied and let* $\widehat{\tau}_n^2 \to^P \tau^2$ *as* $n \to \infty$*. Then, as* $n \to \infty$,

$$T_n(\eta) \to^d \sup_{\eta < t < 1-\eta} \boldsymbol{S}^T(t)\boldsymbol{C}(t)\boldsymbol{C}^{-1}(1)\boldsymbol{C}^0(t)\boldsymbol{S}(t), \quad \eta \in (0, 1/2),$$

where

$$\boldsymbol{S}(t) = \int_0^t \boldsymbol{h}(x)dB(x) - \boldsymbol{C}(t)\boldsymbol{C}^{-1}(1)\int_0^1 \boldsymbol{h}(x)dB(x), \quad t \in [0,1]$$

with $\{B(x),\, x \in [0,1]\}$ *being a Brownian motion and* $\boldsymbol{C}^0(t) = \boldsymbol{C}(1) - \boldsymbol{C}(t)$, $t \in [0,1]$.

Theorem 2 *Let assumptions (A.1)- (A.5) be satisfied and let, as* $n \to \infty$,

$$\widehat{\tau}_n^2 - \tau^2 = o_P((\log\log n)^{-1}).$$

Then

$$\lim_{n\to\infty} P(T_n \le t + b_p(\log n)) = \exp\{-2\exp\{-t/2\}\},\ t \in R^1,$$

where

$$b_p(y) = 2\log y + p\log\log y - 2\log(2^{p/2}\Gamma(p/2)/p), y > 1.$$

Notice that both $T_n(\eta)$ and T_n are asymptotically distribution free under H_0. $T_n(\eta)$ behaves asymptotically as a functional of a Gaussian process, while the limit distribution of T_n is an extreme value type. Approximations for critical values for the test based on T_n can be easily calculated but the convergence is very slow. Concerning approximations of critical values related to $T_n(\eta)$ based on Theorem 1 the limit distributions depend on $\boldsymbol{h}$ and its explicit form is unknown therefore the limit distribution has to be simulated.

There have been derived assertions concerning tests related to $T_n(\eta)$, $\eta \in (0, 1/2)$, under various assumptions. The first papers work with stronger assumptions, see e.g., MacNeill (1978), Jandhyala (1993) and Jandhyala and MacNeill (1989, 1997). These results were later extended to a more general setups, e.g. Bishoff (1998), Kuang (1998), Hansen (2000), Hušková and Picek (2005).

Concerning test procedures based on T_n Albin and Jarušková (2003), Jarušková (1998, 1999) studied limit behavior for independent observations and polynomial trend. Aue et al (2008,2009) extended these results to dependent observations. Theorem 2 above covers all these results as particular cases.

The important issue is the choice of the estimator $\widehat{\tau}_n$ of τ (see Assumption (A.2)). As soon as the error terms $e_1, e_2, \ldots, e_n$ are independent identically distributed with zero mean and finite $(2+\kappa)$th moment with $\kappa > 0$) one can use

$$\widehat{\tau}_n = \frac{1}{n}\sum_{j=1}^n (\widehat{e}_j - \overline{e}_n)^2$$

with

$$\widehat{e}_j = Y_j - \boldsymbol{x}_j^T\widehat{\beta}_n, \quad \overline{e}_n = \frac{1}{n}\sum_{i=1}^n \widehat{e}_i.$$

Usually this estimator has the desired asymptotic properties (formulated as an assumption in Theorem 1 and 2) and also in the finite sample situation it behaves reasonably well. This estimator can be improved by adjusting to a possible change point.

In case of dependent $e_1, \ldots, e_n$ the estimator with flat top kernel can be used:

$$\widehat{\tau}_n^2 = \frac{1}{n}\sum_{j=1}^n (\widehat{e}_j - \overline{e}_n)^2 + \frac{2}{n}\sum_{j=1}^{q_n} w_j \sum_{i=1}^{n-j} (\widehat{e}_i - \overline{e}_n)(\widehat{e}_{i+j} - \overline{e}_n)$$

where

$$w_j = 1I\{1 \le j \le q_n/2\} + 2(1 - j/q_n)I\{q_n/2 < j \le q_n\}.$$

For properly chosen q_n this estimator has also the desired asymptotic property assumed in Theorem 1 and 2 but simulation show that in a finite sample situation one needs larger sample sizes.

It can be shown that both type of test statistics lead to consistent tests under quite general classes of alternatives.

At last we mention that very good approximations for critical values can be obtained by bootstrap. Particularly, circular block bootstrap (see Politis (2003) and Politis and White (2004)) applied to the residuals $\widehat{e}_1, \dots, \widehat{e}_n$ works both asymptotically and also simulation results give satisfactory results.

Further results together some applications will be presented during the talk.

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