1 Introduction

Let $X_1, \ldots, X_n$ be a sequence of independent random variables of distribution function $F$ with $F(1) = 0$, and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$ denote the order statistics of $X_1, \ldots, X_n$ for any fixed $n \geq 1$. Let $k = k(n)$ be sequence of positive integers satisfying:

(K) \quad k \to +\infty, \quad k/n \to 0, \quad \text{and} \quad \log \log n/k \to 0 \quad \text{as} \quad n \to +\infty

We are concerned with this follow stochastic process

\begin{equation}
\frac{1}{s} \log \left( \frac{1}{s} \right) - 1 \log F^{-1}(1 - \left[ \frac{k}{s} \right]/n) - \frac{1}{s} \log F^{-1}(1 - k/n), \quad \text{for} \quad s^2 \geq k/n.
\end{equation}

If $s^2 < k/n$, we put $P_n(s) = 0$. For $s = 1/2$, $P_n(1/2)$ is the so-called Pickands estimator of the extremal index $\gamma$ when $F$ is in the extremal domain of a Generalized Extreme Value Distribution $G_{\gamma}(x) = \exp\left(-\left(1 + \gamma x\right)^{-1/\gamma}\right)$, for $1 + \gamma x > 0$, with $\gamma \in \mathbb{R}$.

This stochastic process has been studied by many authors: Alves (1995), Falk (1994), Pereira (1994), Yun (2000 and 2002), Segers (2002) [8], etc. This latter gave a summarize of the previous works.

The handling of the Pickands process heavily depends that of the stochastic process of the large quantiles $\{X_{n-[k/s]+1,n} \geq 1\}$.

Drees (1995) [4] and de Haan and Ferreira (2006) [3] provided asymptotic uniform and Gaussian approximation of (1.1) when $F$ is in the extremal domain under second order conditions. Segers (2002) [8] used such results to construct new estimators of the extremal index with integrals of statistics on the form of $P_n(s)$. This motivates us to undertake a purely stochastic process approach of the Pickands process in a simpler way but in a more adequate handling in order to derive from this study many potentials applications.

2 Results

Let us introduce some notation. First define the generalized inverse of $F$: $F^{-1}(s) = \inf \{x, F(x) \geq s\}$, $0 < s < 1$, and let

\begin{equation}
p_n(s) = \log(1/s)^{-1}\log F^{-1}(1 - \left[ k/s \right]/n) - F^{-1}(1 - k/n).
\end{equation}

We are going to investigate this Pickands process $\{\kappa_n(s), \frac{k}{n} < s^2 < 1\} = \left\{(\sqrt{k}(P_n(s) - p_n(s)), \frac{k}{n} < s^2 < 1\right\}$.
But it is easy to see that \( P_n(s) \to K(\gamma) = \gamma 1_{(s \neq +\infty)} \) for \( F \in D(G_{1/\gamma}) \). This extends the motivation to the study of \( \{ \kappa^*_n(s), \frac{b}{n} < s^2 < 1 \} = \{ \sqrt{K}(P_n(s) - K(\gamma)), \frac{b}{n} < s^2 < 1 \} \).

Since our conditions depend on auxiliary functions of the representations of functions in the extremal domain \( F \in D(G_{1/\gamma}), \gamma < 0, \gamma > 0, \gamma = +\infty \). For the Frechet case \( \gamma > 0 \), we have

\[
F^{-1}(1-u) = c(1+p(u))u^{-K(\gamma)}\exp\left(\int_u^1 t^{-1}b(t)dt\right).
\]

where \((p(u), b(u)) \to (0,0)\) quand \( u \to 0 \), \( c \) is a positive constant. The representations (2.1) is the Karamata representation.

We fixe two numbers \( a \) and \( b \), \( a < b \), such that \([a, b] \subset [0,1[\). For each \( \gamma > 0 \), we will note \( a(u) = F^{-1}(1-u) \) and for \( \gamma < 0 \), \( a(u) = x_0 - F^{-1}(1-u) \). Set \( k_{n^2} = [a^{-2}]k \) and \( \lambda > 1 \) a real number. Finally we define for an arbitrary function \( h \) defined on \((0,1) \in \mathbb{R},
\]

\[
h_n(\lambda, h, a) = \sup_{0 \leq t \leq \lambda k_{n^2}/n} |h(t)|.
\]

We shall consider the regularity conditions

\[(RC1) \quad \sqrt{k}p_n(\lambda, p, a) \to 0
\]

and

\[(RC2) \quad \sqrt{k}b_n(\lambda, b, a) \to 0.
\]

All unspecified limits occur when \( n \to \infty \). Finally we denote \( o_p(s,a) \) et \( o_p(s,a,b) \) respectively the uniform limits in \( s \in [0,1] \) and \( s \in [a,b] \). Here are our main results.

**Theorem 1** Let \( F \in D(G_{1/\gamma}), \gamma < 0, \gamma > 0, \gamma = +\infty \). Let \( 0 < a < b < 1 \). If (RC1) and (RC2) hold then \( \{\kappa_n(s), s \in [a,b]\} \) converges to a Gaussian process \( \{\mathbb{G}(s), a < s < b\} \) in \( \ell^\infty([a,b]) \), of covariance function

\[
\Gamma(s, t) = \frac{1}{(s-K(\gamma)-1)(t-K(\gamma)-1)\log s\log t}\left\{(s-K(\gamma)-1)t(t-K(\gamma)-1) + t-K(\gamma)K(\gamma)s - K(\gamma)t^2 - K(\gamma)2t^2 + (t-K(\gamma)-1)[K(\gamma)(s-K(\gamma)s - s^2]) + K(\gamma)^2t-K(\gamma)[s-K(\gamma)-s^2]\right\}
\]

with the convention that \( \Gamma(s, t) = \frac{1-s^2}{(\log s)^2(\log t)^2} \), for \( K(\gamma) = 0 \).

Also \( \{\kappa_n(s), a < s < b\} \) converges to \( \mathbb{G} \), in \( \ell^\infty([a,b]) \).

### 3 Proof of the results

It is based on the so-called Hungarian construction of Csörgő et al. (1986) [2]. For this define by \( \{U_n(s), 0 \leq s \leq 1\} \), the uniform empirical distribution function and \( \{V_n(s), 0 \leq s \leq 1\} \), the uniform empirical quantile function, based on the \( n \geq 1 \) first observations \( U_1, U_2, \ldots \) sampled from a uniform random variables on \((0,1) \) and let \( \{\beta_n(s); 0 \leq s \leq 1\} = \{\sqrt{n}(U_n(s) - s), 0 \leq s \leq 1\} \), be the corresponding empirical process. One version of the Csörgő et al. (1986)[2] is the following

**Theorem 2 (Csörgő et al. (1986))** There exists a probability space holding a sequence of independant random variables \( U_1, U_2, \ldots \) uniformly distributed on \((0,1) \) and a sequence of Brownian bridges \( \{B_n(t), 0 \leq t \leq 1\} = \{W_n(t) - tW_n(1), 0 \leq t \leq 1\} \) such that for any \( 0 < \nu < 1/2 \) and \( d \geq 0 \)

\[
\sup_{d/n \leq s \leq 1-d/n} \frac{|\beta_n(s) - B_n(s)|}{s(1-s)^{1/2-\nu}} = O_p(n^{-\nu}).
\]
From this, we derive the Gaussian approximation of the following process \( \{U_{[k/s],n} - \frac{[k/s]}{n}, \frac{k}{n} \leq s \leq 1\} \) in:

**Lemma 1** On the Csörgő et al. (1986) probability space we have

\[
\sup_{k/(n-1) \leq s \leq 1} \left| \sqrt{k} \left( \frac{n}{[k/s]} U_{[k/s],n} - 1 \right) - W_n(1, s) \right| = O_p(1),
\]

where \( W_n(1, s) = s(\frac{k}{n})^{-1/2} W_n(k/(ns)) \) is a Wiener process.

Our proof is performed on the space of Theorem 2. For conciseness, we restrict ourselves to the case \( \gamma > 0 \), since the other cases are proved similarly.

We will just outline the proof. We begin by establishing that

\[
\sqrt{k} \left\{ X_{n-[k/s]+1,n} - F^{-1}(1 - \frac{[k/s]}{n}) \right\} \frac{a(k/n)}{1 + p([k/s]/n)} = -K(\gamma)s^K(\gamma)W_n(1, s) + o_p(s, a, b)
\]

For this, we let

\[ A_n(s) = X_{n-k+1,n} - X_{n-[k/s]+1,n}, \quad B_n(s) = X_{n-[k/s]+1,n} - X_{n-[k/s]^2+1,n} \]

and

\[ a_n(s) = F^{-1}(1 - \frac{k}{n}) - F^{-1}(1 - \frac{[k/s]}{n}), \quad b_n(s) = F^{-1}(1 - \frac{[k/s]}{n}) - F^{-1}(1 - \frac{[k/s]^2}{n}). \]

By using (2.1), we have

\[
X_{n-[k/s]+1,n}/F^{-1}(1 - \frac{[k/s]}{n}) = \frac{1 + p(U_{[k/s],n})}{1 + p([k/s]/n)} \left( \frac{n}{[k/s]} U_{[k/s],n} \right)^{-K(\gamma)} \exp \left( \int_{U_{[k/s],n}}^{[k/s]/n} t^{-1} b(t) dt \right).
\]

We shall treat the three items one by one. Since (RC1) and (RC2) hold, by using Lemma 1 and by the Mean Value Theorem, we arrive at

\[
\sqrt{k} \left( \frac{1 + p(U_{[k/s],n})}{1 + p([k/s]/n)} - 1 \right) \to 0,
\]

in probability, uniformly in \( s \in [a, b] \),

\[
\sqrt{k} \left( \frac{n}{[k/s]} U_{[k/s],n} \right)^{-K(\gamma)} - 1 = -K(\gamma)W_n(1, s)(1 + o_p(s, a, b)) = -K(\gamma)W_n(1, s) + o_p(s, a, b)
\]

and

\[
\sqrt{k} \left\{ \left( \frac{n}{[k/s]} U_{[k/s],n} \right)^{b_n(\lambda, \lambda, a)} - 1 \right\} = b_n(\lambda, \lambda, a)(W_n(1, s) + o_p(s, a, b))(1 + o_p(s, a, b)),
\]

which is \( o_p(s, a, b) \), since \( \sup_{s \in (0,1)} |W_n(1, s)| \) is bounded in probability. Finally

\[
\sqrt{k} \left\{ X_{n-[k/s]+1,n} - F^{-1}(1 - \frac{[k/s]}{n}) \right\} \frac{a([k/s]/n)}{a(k/n)} = -K(\gamma)W_n(1, s) + o_p(s, a, b)
\]

where, for \( a(s) = F^{-1}(1 - s) \), we used the result that \( a([k/s]/n)/a(k/n) \) tends uniformly in \( s^{-K(\gamma)} \) for \( s \in (a, b) \). This achieves the proof of (3.1), thus we use it to prove Theorem 1. We put

\[
W_n(2, s) = (\frac{k}{n})^{-1/2} W_n(k/n), \quad W_n(3, s) = s^2(\frac{k}{n})^{-1/2} W_n(s^{-2} k/n).
\]
This latter is a Gaussian process with covariance function $\min(s^2, t^2)$. We denote now $C_n(s) = \frac{A_n(s)}{B_n(s)}$ and $c_n(s) = \frac{a_n(s)}{b_n(s)}$. We will have
\[
\log C_n(s) - \log c_n(s) = (s^{K(\gamma)} + o_p(a, s)) (C_n(s) - c_n(s))
\]
The same techniques leads to
\[
a_n(s) = a([k/s]/n)(1 - s^{-K(\gamma)})(1 + o_p(s, a)), \quad A_n(s) = a(U_{[k/s], n})(1 - s^{-K(\gamma)})(1 + o_p(s, a))
\]
We determine $b_n(s)$ and $B_n(s)$ in the same way and we conclude that
\[
(\log(1/s))^{-1}\sqrt{k} \{\log C_n(s) - \log c_n(s)\} = (\log(1/s))^{-1} \frac{K(\gamma)}{(1 - s^{-K(\gamma)})} (1 + o_p(s, a))
\]
\[
\times \left\{ (s^{-K(\gamma)} - 1)(W_n(1, s) + o_p(s, a)) + s^{-K(\gamma)}(W_n(2, s) + o_p(s, a)) - W_n(3, s) + o_p(s, a) \right\}
\]
Now by restraining ourselves to $s \in [a, b] \subset [0, 1]$, we get uniformly in those $s$
\[
\kappa_n(s) = \sqrt{k} \{P_n(s) - \log c_n(s)/\log(1/s)\} = G_n(s) + o_p(s, a, b),
\]
where
\[
G_n(s) = \frac{K(\gamma)}{(s^{-K(\gamma)} - 1)\log s} \left\{ (s^{-K(\gamma)} - 1)(W_n(1, s) + s^{-K(\gamma)}W_n(2, s) - W_n(3, s) \right\}
\]
is a Gaussian process with covariance $\Gamma_n(s, t) \rightarrow \Gamma(s, t)$ expressed in Theorem 1. To extend our result to $\kappa^*_n(s)$ we have to prove that under (RC1) and (RC2),
\[
(3.4) \quad \sqrt{k}(p_n(s) - K(\gamma)) \rightarrow 0
\]
In the Theorem 1, the asymptotic laws comes from that of $\frac{n}{k}U_{[k/s], n}$. All the remainder terms are controlled uniformly by the regularity conditions. If we treat (3.4) the corresponding part $\frac{n}{k}[k/s]/n$ satisfies : $\sqrt{k}(\frac{n}{k}[k/s]/n - (\frac{1}{k})K(\gamma)) \rightarrow 0$ uniformly in $s \in [a, b]$. By respectively the same proofs, we arrive at the result for $\gamma < 0, \gamma > 0, \gamma = +\infty$, and this completes the proof. For further more details see Fall, A. M. and Lo, G. S. (2011) [5].

References