On the Pickands stochastic process

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1 Introduction

Let $X_1, \dots, X_n$ be a sequence of independant random variables of distribution function F with $F(1) = 0$, and let $X_{1,n} < X_{2,n}, \dots < X_{n,n}$, denote the order statistics of $X_1, \dots, X_n$ for any fixed $n \geq 1$. Let $k = k(n)$ be sequence of positive integers satisfying:

(K) $$k \to +\infty, \quad k/n \to 0, \quad and \quad \log\log n/k \to 0 \quad as \quad n \to +\infty$$

We are concerned with this follow stochastic process

(1.1) $$P_n(s) = \log(1/s)^{-1} \log \frac{X_{n-k+1,n} - X_{n-[k/s]+1,n}}{X_{n-[k/s]+1,n} - X_{n-[k/s^2]+1,n}}, \quad for \quad s^2 \geq \frac{k}{n}.$$

If $s^2 < \frac{k}{n}$, we put $P_n(s) = 0$. For $s = 1/2$, $P_n(1/2)$ is the so-called Pickands estimator of the extremal index γ when F is in the extremal domain of a Generalized Extreme Value Distribution $G_\gamma(x) = \exp(-(1+\gamma x)^{-\frac{1}{\gamma}}), \quad for \quad 1 + \gamma x > 0$, with $\gamma \in \mathbb{R}$.
This stochastic process has been studied by many authors: Alves (1995), Falk (1994), Pereira (1994), Yun (2000 and 2002), Segers (2002) [8], etc. This latter gave a summarize of the previous works.
The handling of the Pickands process heavily depends that of the stochastic process of the large quantiles$\{X_{n-[k/s]+1,n}, n \geq 1\}$.
Drees (1995) [4] and de Haan and Ferreira (2006)[3] provided asymptotic uniform and Gaussian approximation of(1.1) when F is in the extremal domain under second order conditions. Segers (2002)[8] used such results to construct new estimators of the extremal index with integrals of statistics on the form of $P_n(s)$. This motivates us to undertake a purely stochastic process approach of the Pickands process in a simpler way but in a more adequate handling in order to derive from this study many potentials applications.

2 Results

Let us introduce some notation. First define the generalized inverse of F: $F^{-1}(s) = \inf\{x, F(x) \geq s\}$, $0 < s < 1$, and let

$$p_n(s) = \log(1/s)^{-1} \log \frac{F^{-1}(1 - [k/s]/n) - F^{-1}(1 - k/n)}{F^{-1}(1 - [k/s^2]/n) - F^{-1}(1 - [k/s]/n)}.$$

We are going to investigate this Pickands process $\left\{\kappa_n(s), \frac{k}{n} < s^2 < 1\right\} = \left\{\sqrt{k}(P_n(s) - p_n(s)), \frac{k}{n} < s^2 < 1\right\}$

But it is easy to see that $P_n(s) \to K(\gamma) = \gamma \mathbf{1}_{\{\gamma \neq +\infty\}}$ for $F \in D(G_{1/\gamma})$. This extends the motivation to the study of $\{\kappa_n^*(s), \frac{k}{n} < s^2 < 1\} = \left\{\sqrt{k}(P_n(s) - K(\gamma)), \frac{k}{n} < s^2 < 1\right\}$.
Since our conditions depend on auxilliary functions of the representations of functions in the extremal domain $F \in D(G_{1/\gamma})$, $\gamma < 0$, $\gamma > 0$, $\gamma = +\infty$. For the Frechet case $\gamma > 0$, we have

$$(2.1) \qquad F^{-1}(1-u) = c(1+p(u))u^{-K(\gamma)} \exp\left(\int_u^1 t^{-1}b(t)dt\right).$$

where $(p(u), b(u)) \to (0,0)$ quand $u \to 0$, c is a positive constant The representations (2.1) is the Karamata representation.
We fixe two numbers a and b, $a < b$, such that $[a,b] \subset]0,1[$. For each $\gamma > 0$, we will note $a(u) = F^{-1}(1-u)$ and for $\gamma < 0$, $a(u) = x_0 - F^{-1}(1-u)$. Set $k_{a^2} = [a^{-2}]k$ and $\lambda > 1$ a real number. Finally we define for an arbitrary function h defined on $(0,1)$ in $\mathbb{R}$,

$$h_n(\lambda, h, a) = \sup_{0 \le t \le \lambda k_{a^2}/n} |h(t)|.$$

We shall consider the regularity conditions

$$(\text{RC1}) \qquad \sqrt{k} p_n(\lambda, p, a) \to 0$$

and

$$(\text{RC2}) \qquad \sqrt{k} b_n(\lambda, b, a) \to 0.$$

All unspecified limits occur when $n \to \infty$. Finally we denote $o_p(s,a)$ et $o_p(s,a,b)$ respectively the uniform limits in $s \in [0,1]$ and $s \in [a,b]$. Here are our main results.

Theorem 1 *Let $F \in D(G_{1/\gamma})$, $\gamma < 0$, $\gamma > 0$, $\gamma = +\infty$. Let $0 < a < b < 1$. If $(RC1)$ and $(RC2)$ hold then $\{\kappa_n(s), s \in [a,b]\}$ converges to a Gaussian process $\{\mathbb{G}(s), a < s < b\}$ in $\ell^\infty([a,b])$, of covariance function*

$$\Gamma(s,t) = \frac{1}{(s^{-K(\gamma)}-1)(t^{-K(\gamma)}-1)\log s \log t}\{(s^{-K(\gamma)}-1)[t(t^{-K(\gamma)}-1) + t^{-K(\gamma)}K(\gamma)s - K(\gamma)t^2 - K(\gamma)^2t^2] + (t^{-K(\gamma)}-1)[K(\gamma)(s^{-K(\gamma)}t - s^2)] + K(\gamma)^2 t^{-K(\gamma)}[s^{-K(\gamma)} - s^2]\}$$

with the convention that $\Gamma(s,t) = \frac{1-s^2}{(\log s)^2(\log t)^2}$, *for* $K(\gamma) = 0$.
Also $\{\kappa_n^*(s), a < s < b\}$ converges to $\mathbb{G}$, in $\ell^\infty([a,b])$.

3 Proof of the results

It is based on the so-called Hungarian construction of Csörgő et al. (1986) [2]. For this define by $\{U_n(s), 0 \le s \le 1\}$, the uniform empirical distribution function and $\{V_n(s), 0 \le s \le 1\}$, the uniform empirical quantile function, based on the $n \ge 1$ first observations $U_1, U_2, \ldots$ sampled from a uniform random variables on $(0,1)$ and let $\{\beta_n(s); 0 \le s \le 1\} = \{\sqrt{n}(U_n(s) - s), 0 \le s \le 1\}$, be the corresponding empirical process. One version of the Csörgő et al. (1986)[2] is the following

Theorem 2 (Csörgő et al. (1986)) *There exists a probability space holding a sequence of independant random variables $U_1, U_2, \ldots$ uniformly distributed on $(0,1)$ and a sequence of Brownian bridges $\{B_n(t), 0 \le t \le 1\} = \{W_n(t) - tW_n(1), 0 \le t \le 1\}$ such that for any $0 < \nu < 1/2$ and $d \ge 0$*

$$\sup_{d/n \le s \le 1-d/n} \frac{|\beta_n(s) - B_n(s)|}{\{s(1-s)\}^{1/2-\nu}} = O_p(n^{-\nu}).$$

From this, we derive the Gaussian approximation of the following process $\{U_{[k/s],n} - \frac{[k/s]}{n}, \frac{k}{n} \leq s \leq 1\}$ in:

Lemma 1 *On the Csörgő et al. (1986) probability space we have*

$$\sup_{k/(n-1) \leq s \leq 1} \left| \sqrt{k}(\frac{n}{[k/s]} U_{[k/s],n} - 1) - W_n(1,s) \right| = O_p(1),$$

where $W_n(1,s) = s(\frac{k}{n})^{-1/2} W_n(k/(ns))$ *is a Wiener process.*

Our proof is performed on the space of Theorem 2. For conciseness, we restrict ourselves to the case $\gamma > 0$, since the other cases are proved similary.

We will just outline the proof. We begin by establishing that

$$\frac{\sqrt{k}\left\{X_{n-[k/s]+1,n} - F^{-1}(1 - \frac{[k/s]}{n})\right\}}{a(k/n)} = -K(\gamma) s^{K(\gamma)} W_n(1,s) + o_p(s,a,b) \tag{3.1}$$

For this, we let
$A_n(s) = X_{n-k+1,n} - X_{n-[k/s]+1,n}$, $B_n(s) = X_{n-[k/s]+1,n} - X_{n-[k/s^2]+1,n}$ and
$a_n(s) = F^{-1}(1 - \frac{k}{n}) - F^{-1}(1 - \frac{[k/s]}{n})$, $b_n(s) = F^{-1}(1 - \frac{[k/s]}{n}) - F^{-1}(1 - \frac{[k/s^2]}{n})$.

By using (2.1), we have

$$X_{n-[k/s]+1,n}/F^{-1}(1 - \frac{[k/s]}{n}) = \frac{1 + p(U_{[k/s],n})}{1 + p([k/s]/n)} (\frac{n}{[k/s]} U_{[k/s],n})^{-K(\gamma)} \exp(\int_{U_{[k/s],n}}^{[k/s]/n} t^{-1} b(t) dt).$$

We shall treat the three items one by one. Since $(RC1)$ and $(RC2)$ hold, by using Lemma 1 and by the Mean Value Theorem, we arrive at

$$\sqrt{k}(\frac{1 + p(U_{[k/s],n})}{1 + p([k/s]/n)} - 1) \to 0, \tag{3.2}$$

in probability, uniformly in $s \in [a,b]$,

$$\sqrt{k}(\frac{n}{[k/s]} U_{[k/s],n})^{-K(\gamma)} - 1 = -K(\gamma) W_n(1,s)(1 + o_p(s,a,b)) = -K(\gamma) W_n(1,s) + o_p(s,a,b) \tag{3.3}$$

and

$$\sqrt{k}\left\{(\frac{n}{[k/s]} U_{[k/s],n})^{b_n(\lambda,b,a)} - 1\right\} = b_n(\lambda,b,a)(W_n(1,s) + o_p(s,a,b))(1 + o_p(s,a,b)),$$

which is $o_p(s,a,b)$, since $\sup_{s \in (0,1)} |W_n(1,s)|$ is bounded in probability. Finally

$$\frac{\sqrt{k}\left\{X_{n-[k/s]+1,n} - F^{-1}(1 - \frac{[k/s]}{n})\right\}}{a([k/s]/n)} = -K(\gamma) W_n(1,s) + o_p(s,a,b)$$

where, for $a(s) = F^{-1}(1-s)$, we used the result that $a([k/s]/n)/a(k/n)$ tends uniformly in $s^{-K(\gamma)}$ for $s \in (a,b)$. This achieves the proof of (3.1), thus we use it to prove Theorem 1. We put

$$W_n(2,s) = (\tfrac{k}{n})^{-1/2} W_n(k/n),\ W_n(3,s) = s^2 (\tfrac{k}{n})^{-1/2} W_n(s^{-2}k/n).$$

This latter is a Gaussian process with covariance function $\min(s^2, t^2)$. We denote now $C_n(s) = \frac{A_n(s)}{B_n(s)}$ and $c_n(s) = \frac{a_n(s)}{b_n(s)}$. We will have

$$\log C_n(s) - \log c_n(s) = (s^{K(\gamma)} + o_p(a, s))\ (C_n(s) - c_n(s))$$

The same techniques leads to

$$a_n(s) = a([k/s]/n))(1 - s^{-K(\gamma)})(1 + o_p(s, a)),\ A_n(s) = a(U_{[k/s],n})(1 - s^{-K(\gamma)})(1 + o_p(s, a))$$

We determine $b_n(s)$ and $B_n(s)$ in the same way and we conclude that

$$(\log(1/s))^{-1}\sqrt{k}\,\{\log C_n(s) - \log c_n(s)\} = \log(1/s)^{-1}\frac{K(\gamma)}{(1 - s^{-K(\gamma)})}(1 + o_p(s, a))$$

$$\times \left\{(s^{-K(\gamma)} - 1)(W_n(1, s) + o_p(s, a)) + s^{-K(\gamma)}(W_n(2, s) + o_p(s, a)) - W_n(3, s) + o_p(s, a)\right\}.$$

Now by restraining ourselves to $s \in [a, b] \subset]0, 1[$, we get uniformly in those s,

$$\kappa_n(s) = \sqrt{k}\,\{P_n(s) - \log c_n(s)/\log(1/s)\} = \mathbb{G}_n(s) + o_p(s, a, b),$$

where

$$\mathbb{G}_n(s) = \frac{K(\gamma)}{(s^{-K(\gamma)} - 1)\log s}\left\{(s^{-K(\gamma)} - 1)(W_n(1, s) + s^{-K(\gamma)}W_n(2, s) - W_n(3, s)\right\}$$

is a Gaussian process with covariance $\Gamma_n(s, t) \to \Gamma(s, t)$ expressed in Theorem 1. To extend our result to $\kappa_n^*(s)$ we have to prove that under $(RC1)$ and $(RC2)$,

$$\sqrt{k}(p_n(s) - K(\gamma)) \to 0 \qquad (3.4)$$

In the Theorem 1, the asymptotic laws comes from that of $\frac{n}{k}U_{[k/s],n}$. All the remainder terms are controlled uniformly by the regularity conditions. If we treat (3.4) the corresponding part $\frac{n}{k}[k/s]/n$ satisfies : $\sqrt{k}(\frac{n}{k}[k/s]/n - (\frac{1}{s})^{K(\gamma)}) \to 0$ uniformly in $s \in [a, b]$. By respectively the same proofs, we arrive at the result for $\gamma < 0$, $\gamma > 0$, $\gamma = +\infty$, and this completes the proof. For further more details see Fall, A. M. and Lo, G. S. (2011) [5].

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