DIFFICULTIES IN THE IMPLEMENTATION OF THE CONTROL CHART FOR THE COEFFICIENT OF VARIATION WITH GAMMA OBSERVATIONS

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ABSTRACT

The coefficient of variation γ is defined as the ratio of the population standard deviation to the population mean. It is often estimated by W, the ratio of the sample standard deviation to the sample mean.

The distribution of W depends on the distribution of the original variables. Even under normality, it is difficult to obtain.

Several authors studied the distribution when γ is lower than 0.30 and the sample size is no more than 15. The validity of these distributions for certain values of γ and subgroups sample sizes, strongly limits the applications of control charts. With greater subgroup sizes, the asymptotic distribution would be valid in a broader context.

For asymmetric variables such as Gamma, some transformations of the variables have been studied, being necessary to specify the value of a pivotal quantity z to obtain the control limits.

Baek et alt provide a table with different values of z, but they do not indicate their relationship with the parameters of the Gamma distribution. Furthermore, when they study the efficiency of the control chart, they consider shape parameters which are near to normality.

The authors of this paper investigate the relationship between z and the parameters of Gamma distribution, finding out that it only depends on the shape parameter. Gamma distributions are considered with shape parameters equal to 25; 16 and 11 which correspond to γ values of 0.20; 0.25 and 0.30 respectively. Efficiency studies are conducted using the Average Run Length (ARL). Control limits are calculated. It is found that the ARL curves exhibit a higher frequency of false alarms.

The performance of this strategy is analized when the asymmetric distribution is ignored and normal limits are used. In this situation the ARL curves show an ARL value greater than expected in the under control situation, and a very slow decrease of the ARL in the out of control situations, which reflects low efficiency.

KEY WORDS: *Coefficient of variation, Control charts, Gamma Distribution, ARL*

1. INTRODUCTION

The main objective of Statistical Process Control (SPC) is to control the variability. The traditional Shewhart control charts by variables, ($\overline{X}$, *R*) or ($\overline{X}$, S) are a very useful tool for this purpose. The $\overline{X}$ chart is used to monitor changes in the mean and the *R* or *S* control chart are used to monitor the variability of the process.

These charts are useful in many scenarios as long as the process under control is characterized by having stable mean and variance. This is an essential property and only in these cases the $\overline{X}$ chart can be used to monitor the process average and range chart or deviations to monitor the variability. But sometimes it is natural that the variability does not remain constant and fluctuates depending on the average, so the above charts are not suitable. In this case a control chart for the coefficient of variation (CV) would be appropriate to use. Assuming that the standart deviation is proportional to the mean, the CV should be kept constant.

It is necessary to know the distribution of sample coefficient of variation to design the chart. Several authors have derived this distribution under normal random variables (Hendricks, W. and Robey, W. (1936); Iglewicz, B. and Myers, R. (1970), Chang W. Kang et alt (2007) and Forkman, J . and Verrill, S. (2008)) and asymmetric variables like Gamma (Linhart, H. (1965), Jae-Won Baek et alt. (2006)).

In many practical situations we deal with nonnegative variables; in other normal distribution can not be considered because it does not fits. In such cases other distributions such as Log-normal and Gamma can be used. Both of them have two parameters. The advantage of the Log-normal distribution is that normal theory can be applied once transformed the variable. When transformed variables are not of interest, the Gamma distribution may be an appropriate choice. A two-parameter Gamma model is flexible enough in practice to adjust positive random variables. Under this distribution the control chart for the CV requires more complex calculations to obtain the control limits.

According to previous studies[1] the control chart for the CV using normal limits is not robust to departures from normality, even if they are moderate, so in these cases the use of those limits would not be advisable.

The main purpose of this paper is to analyze the properties of the control chart for the CV when very asymmetric Gamma distributions are under consideration.

2. OBJECTIVES

-Analyze the difficulties of the implementation of the control chart for the CV in certain scenarios in which very asymmetric variables type Gamma are considered.

-Study by simulation procedures the performance of the chart when particular limits calculated under Gamma variables are used.

-Analyze the behavior of this strategy when asymmetric distribution of the original variables is ignored and normal limits are used.

3. METHODOLOGY

A Shewhart control chart for the CV contains three lines: a center line, and the upper and lower control limits. These limits are set at values that are exceeded with some known probability. The center line is set at the population CV (γ) value, because the moments of the sample coefficient of variation *W* are infinite. Distribution of W must be known to obtain the upper and lower control limits.

Let $X_1, X_2, \ldots X_n$ random variables with density function Gamma $f(x,\lambda,\alpha)$, given by

$$f(x,\lambda,\alpha) = \frac{1}{\Gamma(\frac{\lambda}{2})(2\alpha)^{\frac{\lambda}{2}}} x^{(\frac{\lambda}{2})-1} . e^{-\frac{x}{2\alpha}} \quad x > 0, \lambda > 0, \alpha > 0$$

, where 2α and $\frac{\lambda}{2}$ are scale and shape parameters respectively, and $E(X) = \lambda\alpha$, $Var(X) = 2\alpha^2\lambda$ and $\gamma = \frac{\sigma}{\mu} = \sqrt{\frac{2}{\lambda}}$.

Cohen y Whitten (1988); Bowman, Shenton y col (1988); Balakrishnan y Cohen (1991) provide a detailed analysis of the maximum likelihood estimators and of those obtained by moment estimation method, of the Gamma distribution parameters.

The likelihood estimator of γ, taking into account the invariance property is:

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[1] [1] Barbiero,C; Flury,M; Pagura,J; Quaglino,M y Ruggieri, M. (2009) Control Chart for the Coefficient of variation. 57 th Session of the ISI. Durban. South Africa. CPM 110

$$\hat{\gamma}_{MV} = W = \sqrt{\frac{2}{\hat{\lambda}_{MV}}} = \sqrt{\frac{2}{z^{-1}+\frac{1}{3}}}$$

where $Z = [\ln \bar{X} - \overline{\ln X}\,] \sim \{\frac{[1+\frac{(1+1/n)}{3\lambda}]}{n\lambda}\}\chi_{n-1}{}^{2}$ (Johnson & Kotz (1994)), so the density probability function of the random variable *W*, is obtained as follows:

$$G_w(w) = P\{W \le w\} = P\left\{\sqrt{\frac{2}{z^{-1}+\frac{1}{3}}} \le w\right\} = P\left\{\frac{\sqrt{2}}{w} \le (z^{-1}+\frac{1}{3})^{0,5}\right\}$$

$$G_w(w) = P\{Z \le \frac{3w^2}{6-w^2}\} = F_z(\frac{3w^2}{6-w^2}) .$$

$$g(w) = G'(w) = f_z(\frac{3w^2}{6-w^2}).\frac{36w}{(6-w^2)^2};\ \text{para } |w| < \sqrt{6}$$

The control limits for the CV chart are obtained as follows:

$$P\{W \le UCL\} = 1-\frac{\alpha}{2} \qquad (1)$$

$$P\{W \le CL\} = \frac{1}{2} \qquad (2)$$

$$P\{W \le LCL\} = \frac{\alpha}{2} \qquad (3)$$

The distribution of W is used to obtain Average Run Length (ARL) curves to evaluate the performance of the chart, under control and out of control situations.

Jae-Won Baek et alt(2006) provide the following control-limits using (1), (2) and (3):

$$UCL = \left\{\gamma^2\left[\frac{\lambda_L \chi^2_{n-1,1-\alpha/2}}{z(n(z^{-1}+\frac{1}{3})+2}\right]\right\}^{1/2} \quad with\ \lambda_L = \frac{1}{2}\left\{1+[1+\frac{4(n+1)z}{3\chi^2_{n-1,1-\alpha/2}}\ ]^{1/2}\right\}$$

$$CL = \left\{\gamma^2\left[\frac{\lambda_C \chi^2_{n-1;0,5}}{z(n(z^{-1}+\frac{1}{3})+2}\right]\right\}^{1/2} \quad \text{with} \quad \lambda_C = \frac{1}{2}\left\{1+[1+\frac{4(n+1)z}{3\chi^2_{n-1;0,5}}\ ]^{1/2}\right\}$$

$$LCL = \left\{\gamma^2\left[\frac{\lambda_U \chi^2_{n-1,\alpha/2}}{z(n(z^{-1}+\frac{1}{3})+2}\right]\right\} \quad \text{with} \quad \lambda_U = \frac{1}{2}\left\{1+[1+\frac{4(n+1)z}{3\chi^2_{n-1,\alpha/2}}\ ]^{1/2}\right\}$$

4. DIFFICULTIES IN THE DESIGN OF THE CONTROL CHART FOR THE CV.

The first difficulty that arises is to choose the adequate z value. Baek et alt.(2006) provide a table with different values of z, but they do not indicate their relationship with the scale and shape parameters of the underlying distribution of the variables.

Researching about this relationship, the authors of this study found that these values depend only on the shape parameter. When this parameter decreases, the z values increase (Table 1).

Tabla 1
Z values for different shape parameters (α = 0.0027)

	Shape parameter ($\frac{\lambda}{2}$)					
	400	**100**	**44**	**25**	**16**	**11**
Coefficient of variation $\gamma = \sqrt{\frac{2}{\lambda}}$	0.050	0.100	0.150	0.200	0.250	0.300
Pivotal Quantity (z)	0.001	0.004	0.010	0.020	0.030	0.040

Settings with remarkably asymmetric Gamma distribution are chosen, with shape parameter equal to 25, 16 and 11 that correspond to γ values of 0.20, 0.25 and 0.30 respectively. The papers about efficiency of control-chart with Gamma variables only considered distribution with big shapes parameters, becoming as Normality.

Once selected the z values, the second difficulty is to obtain the control limits for each particular case associated with some probability of false alarm. In this case was selected P (e_I) = 0.0027. They are not tabulated and require complex calculations. Following the proposal of Baek et alt (2006)., we calculate the control chart limits for different subgroup sizes of n.

Table 2
Control limits under Gamma distribution for different γ and n values (α = 0.0027)

γ	Subgroup size	
	n = 5	**n = 10**
γ = 0.20 (z = 0.02)	UCL = 0.3817 LCL = 0.0205	UCL = 0.3352 LCL = 0.0782
γ = 0.25 (z = 0.03)	UCL = 0.4779 LCL = 0.0266	UCL = 0.4203 LCL = 0.0996
γ = 0.30 (z = 0.04)	UCL = 0.5739 LCL = 0.0329	UCL = 0.5054 LCL = 0.1213

To compare the performance of the Gamma control chart with the Normal control chart , were calculated normal limits using the same values of γ and subgroups sizes of n (Table 3).

Table 3
Control limits under Normal distribution for different γ* and *n* values *(α = 0.0027)

	Subgroup size

γ	**n = 5**	**n = 10**
γ = 0.20	UCL = 0.4489 LCL = 0.0321	UCL = 0.3668 LCL = 0.0733
γ = 0.25	UCL = 0.6000 LCL = 0.0396	UCL = 0.4600 LCL = 0.0901
γ = 0.30	UCL = 0.7345 LCL = 0.0476	UCL = 0.5696 LCL = 0.1083

The differences between the calculated limits under both distributions are stronger for small subgroup sizes.

5. EFFICIENCY COMPARISON OF CONTROL CHARTS WITH GAMMA VARIABLES

ARL curves are obtained by simulation procedures in order to compare the behavior of the chart when limits are set according to Gamma or Normal distribution. Each point of the curve is the average of 10000 iterations of the variable run length. Different scenarios were considered with subgroup sizes of 5 and 10 and γ equal to 0.20; 0.25 and 0.30 in the under control situations. To generate out of control observations, successive increases of 4% of γ were fixed. The results are shown in Table 4.

Table 4
ARL with Gamma and Normal limits

Increase (% γ)	**Gamma limits**						**Normal limits**					
	γ **= 0.20**		γ **= 0.25**		γ **= 0.30**		γ **= 0.20**		γ **= 0.25**		γ **= 0.30**	
	n = 5	n = 10	n = 5	n = 10	n = 5	n =10	n = 5	n = 10	n = 5	n = 10	n = 5	n =10
0	166.1	184.2	167.8	172.7	173.3	162.7	551.1	486.1	680.1	546.1	776.5	665.8
4	107.4	123.3	115.7	126.2	110.5	114.5	471.3	374.8	677.2	425.4	780.0	575.9
8	71.0	81.7	75.1	83.0	75.9	81.9	374.4	224.1	599.4	293.5	714.9	399.8
12	51.6	51.4	51.0	49.9	52.3	53.0	254.4	139.4	482.6	168.9	560.3	253.8
16	36.1	33.8	38.3	35.7	42.1	35.1	174.4	90.5	378.0	105.3	433.5	164.5
20	28.5	23.3	29.0	23.1	29.4	24.7	126.1	58.4	264.6	73.3	325.9	97.8
24	21.1	17.4	22.0	16.7	22.6	18.6	96.3	37.5	183.1	48.8	228.0	66.4
28	17.1	13.1	16.6	13.3	17.7	13.6	67.7	26.0	129.2	32.8	165.6	47.0
/32	13.9	10.0	14.7	10.1	13.8	10.0	50.2	19.7	97.4	25.3	122.6	33.1
36	10.8	7.7	12.2	8.4	12.2	8.5	36.8	14.2	75.5	17.3	89.1	24.2
40	9.7	6.3	9.6	6.9	10.5	6.9	30.9	11.0	54.0	13.1	69.3	17.9
44	7.7	5.1	8.3	5.3	8.5	5.6	23.6	8.7	43.4	10.8	54.8	14.2
48	7.1	4.4	7.0	4.3	7.4	4.7	20.2	7.1	33.5	8.8	41.5	10.9

In the under control situation, an ARL value of 370 is expected for a probability of false alarm of 0.0027. From Table 4 (Gamma limits) we can see that the values obtained in such situation are lower, which indicates a significant and unwanted increase in the rate of false alarm. For Normal limits we can see that the ARL is higher than fixed (between 1.5 and 2 times the ARL value) in the under control situation and in out of control settings the curves show a very slow decrease in the ARL values, which reflects the low efficiency of the chart.

6. CONCLUSIONS

The results presented in Section 5 of this paper, demonstrate the lack of robustness of the Shewhart control chart for the CV with highly asymmetric Gamma variables, when limits are set ignoring the original distribution of the variables. If using normal limits, the desired under control ARL is not obtained and in the out of control situations the ARL decrease very slowly. When considering specific limits, the performance is better in out of control situations, but the false alarm rate is higher than fixed, in the studied scenarios. On the other hand, the design of the chart depends on a variety of considerations (subgroup size, degree of asymmetry defined by the shape parameter and the adequate choice of the pivotal quantity z), being not easy to implement this strategy on the ground.

These considerations lead to the need for further research on this subject, in order to obtain a chart with the desired performance.

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