On Recovering Lost Information for Sequential Estimation in a Uniform Distribution

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1. Introduction

Let us begin by assuming availability of a sequence $X_1, X_2, ...$ of independent observations following a $U(0, \theta)$ population with an unknown scale parameter $\theta, 0 < \theta < \infty$. Having recorded $X_1, ..., X_n$, let us denote the customary estimators:

$$\begin{array}{ll} \text{Sample Mean:} & \overline{X}_n \equiv n^{-1}\Sigma_{i=1}^{n} X_i \\ \text{Sample Maximum:} & X_{n:n} \equiv \max\{X_1, ..., X_n\}. \end{array} \quad (1.1)$$

Ghosh and Mukhopadhyay (1975) introduced a purely sequential minimum risk point estimation procedure for θ (> 0). This was developed under a squared error loss plus a linear cost function of sampling. Mukhopadhyay et al. (1983) broadened that earlier methodology considerably. In these papers, the unknown parameter θ was estimated by the associated randomly stopped $X_{n:n}$ in both the loss function and the stopping rule.

Subsequently, Mukhopadhyay (1987) pursued a slightly different idea for sequential minimum risk point estimation for θ. He used the associated randomly stopped versions of $X_{n:n}$ or $2\overline{X}_n$ in either the loss function or the stopping rule. Performances of such procedures were compared with those associated with the earlier proposed sequential estimators of θ based on $X_{n:n}$.

But, clearly, using a randomly stopped version of $2\overline{X}_n$ would amount to some loss of information when compared with a corresponding randomly stopped $X_{n:n}$ in both the loss function and the stopping rule. In this note, we explore some novel approaches for recovering such lost information by fine-tuning the loss function itself and then tailoring the associated sequential methodologies appropriately.

We will examine how the sequential risks of our newly proposed methodologies would compare with those associated with the existing sequential estimators. We will also present small, moderate as well as large sample-size performances of the new randomly stopped versions of $X_{n:n}$ and explore some selected second-order properties.

Designs of original two-stage and multi-stage sampling methodologies and their practical implementations in large-scale sample surveys date back to Mahalanobis's (1940) pioneering research. The broad area of multi-stage and sequential estimation problems may be reviewed from Sen (1981), Woodroofe (1982), Siegmund (1985), Mukhopadhyay and Solanky (1994), Ghosh and Sen (1990), Ghosh et al. (1997), Mukhopadhyay et al. (2004), and Mukhopadhyay and de Silva (2009) among other sources.

Having recorded $X_1, ..., X_n$, let us consider a generic estimator U_n of θ and let the loss function associated with this estimator be:

$$L_n\left(T_n, \theta\right) = K\left(U_n - \theta\right)^2 + cn, \tag{1.1}$$

where $c(>0)$ is the cost per unit sample and $K(>0)$ is some appropriate weight. In this paper, we will consider two rival unbiased estimators of θ and two different weights in their respective loss functions. Then, we consider tuning these weights involved in these loss functions in such a way that the (asymptotic) risks associated with them to become comparable.

This adjustment in the loss function becomes particularly relevant when one estimator of θ is based on the complete sufficient statistic $X_{n:n} = \max\{X_1, ..., X_n\}$ and the rival estimator is based on a non-sufficient statistic, say, $\overline{X}_n$. Now, the risk associated with the loss function given in (1.1) is given by:

$$R_n(c;\theta) \equiv E_\theta\left[L_n\left(U_n, \theta\right)\right] = KE_\theta\left[\left(U_n - \theta\right)^2\right] + cn. \tag{1.2}$$

For the two choices of estimators of θ, namely $U_n^{(1)}$ and $U_n^{(2)}$, let the weights in the corresponding loss functions be respectively K_1 and K_2. Then, after equating the two risks we should have:

$$K_1 E_\theta\left[\left(U_n^{(1)} - \theta\right)^2\right] = K_2 E_\theta\left[\left(U_n^{(2)} - \theta\right)^2\right]. \tag{1.3}$$

In this way, one will come up with two different loss functions. In general, K_1 and K_2 may involve our unknown parameter θ. This approach can motivate many customary weighted loss functions that are often employed under decision theory.

To be more specific, we may fix $U_n^{(1)} \equiv \left(\frac{n+1}{n}\right)X_{n:n}$ and $U_n^{(2)} \equiv 2\overline{X}_n$, the two possible choices for the final estimator of θ. The respective risks will be given by:

$$\begin{aligned} R_n^{(1)}(c;\theta) \equiv E_\theta\left[L_n\left(U_n^{(1)}, \theta\right)\right] &= \tfrac{K_1\theta^2}{n(n+2)} + cn \approx \tfrac{K_1\theta^2}{n^2} + cn, \text{ for large } n, \\ \text{and } R_n^{(2)}(c;\theta) \equiv E_\theta\left[L_n\left(U_n^{(2)}, \theta\right)\right] &= \tfrac{K_2\theta^2}{3n} + cn. \end{aligned} \tag{1.4}$$

Now, employing (1.3) in conjunction with (1.4) yields (for large n):

$$K_2 = 3n^{-1}K_1.$$

Thus, our two loss functions now become:

$$L_n^{(1)}\left(U_n^{(1)}, \theta\right) = A\left(U_n^{(1)} - \theta\right)^2 + cn, \tag{1.5}$$

and

$$L_n^{(2)}\left(U_n^{(2)}, \theta\right) = \frac{3A}{n}\left(U_n^{(2)} - \theta\right)^2 + cn, \tag{1.6}$$

with the asymptotic risk for both:

$$R_n^{(2)}(c;\theta) \approx R_n^{(1)}(c;\theta) = \frac{A\theta^2}{n^2} + cn. \tag{1.7}$$

This risk is minimized when

$$n \text{ is the smallest integer} \geq \left(2A\theta^2/c\right)^{1/3} = n^*, \text{ say.} \tag{1.8}$$

The minimum risk associated with (1.7)-(1.8) is given by:

$$R_{n^*}^{(1)}(c;\theta) = R_{n^*}^{(2)}(c;\theta) \approx \frac{A\theta^2}{n^{*2}} + cn^* = \frac{cn^{*3}}{2n^{*2}} + n^* \approx \frac{3}{2}cn^*. \tag{1.9}$$

The required optimal fixed sample size n^* from (1.8) depends on θ and hence may be estimated or updated step-by-step determined sequentially. In section 2, we will specifically indicate how to implement such sequential methodologies. From (1.8), we note that we have $n^* = \left(2A\theta^2/c\right)^{1/3}$. Thus, in general, we may proceed as follows:

We may begin with pilot data $X_1, ..., X_m$ of size $m(\geq 1)$ and then move forward by taking one additional observation at-a-time, if needed, according to a purely sequential stopping time defined as follows: Let us denote:

$$N = \inf\left\{n \geq m(\geq 1) : n \geq (2A/c)^{1/3}\, T_n^{2/3}\right\}. \tag{1.10}$$

Here, the consistent (for θ) estimator T_n that is used in defining the boundary crossing may look rather different from $U_n^{(1)}$ or $U_n^{(2)}$. We will introduce alternative choices in Section 2. But, once sampling would stop according to the stopping rule (1.10), we would be accruing the final dataset $\{N, X_1, ..., X_m, ..., X_N\}$, and our final estimator of θ at termination will be either $U_N^{(1)}$ or $U_N^{(2)}$.

In Section 2, we will propose four different stopping rules $N_1 - N_4$ based on four different choices of the estimator T_n. However, the final estimator of θ will be either $U_{N_j}^{(1)}$ or $U_{N_j}^{(2)}, j = 1,2,3,4$. Considering N_j as our stopping time and $U_N \equiv U_{N_j}^{(i)}$, the associated risk will be the expected value of $L_{N_j}^{(i)}\left(U_{N_j}^{(i)}, \theta\right)$, $j = 1,2,3,4$ and $i = 1,2$.

Our goal is to compare the risks associated with the corresponding sequential versions of the fixed-sample-size unbiased estimators $2\overline{X}_n$ that is based on non-sufficient statistic with $\left(\frac{n+1}{n}\right)X_{n:n}$ that is based on a complete sufficient statistic. A similar idea was initially proposed by Mukhopadhyay (1987) in a much smaller scale. Section 3 presents some relevant data analyses.

2. Four Sequential Methodologies and the Stopping Times

The previous section introduced a generic sequential stopping rule in (1.10). We may contemplate using different expressions of T_n or T_n^2 or $T_n^{2/3}$ in defining the boundary condition found in (1.10) as different estimators of θ, θ^2, or $\theta^{2/3}$ respectively. Let us consider the following choices:

(i) $T_n^{(1)} = X_{n:n}$: Estimating θ based on the complete sufficient statistic, $X_{n:n}$;

(ii) $T_n^{(2)} = \left(\frac{n+1}{n}\right) X_{n:n}$: Estimating θ unbiasedly based on the complete sufficient statistic $X_{n:n}$;

(iii) $T_n^{(3)} = 4\left(1 + \frac{1}{3n}\right)^{-1} \overline{X}_n^2$: Estimating θ^2unbiasedly based on the non-sufficient statistic, $\overline{X}_n$;

(iv) $T_n^{(4)} = \left(\frac{3n}{3n+2}\right) X_{n:n}^{2/3}$: Estimating $\theta^{2/3}$ unbiasedly based on the complete sufficient statistic, $X_{n:n}$

One should note that $T_n^{(2)}$ coincides with $U_n^{(1)}$.

Now, given these four estimators we go ahead and propose four different stopping rules as follows:

$$N_1 = \inf\left\{n \geq m(\geq 1) : n \geq (2A/c)^{1/3}\left(T_n^{(1)}\right)^{2/3}\right\}; \tag{2.1}$$

$$N_2 = \inf\left\{n \geq m(\geq 1) : n \geq (2A/c)^{1/3}\left(T_n^{(2)}\right)^{2/3}\right\}; \tag{2.2}$$

$$N_3 = \inf\left\{n \geq m(\geq 1) : n \geq (2A/c)^{1/3}\left(T_n^{(3)}\right)^{1/3}\right\}; \tag{2.3}$$

$$N_4 = \inf\left\{n \geq m(\geq 1) : n \geq (2A/c)^{1/3}\, T_n^{(4)}\right\}. \tag{2.4}$$

We can show that $P_\theta(N_j < \infty) = 1, j = 1, 2, 3, 4$, and hence upon termination of the stopping time N_j, the unknown parameter θ will be finally estimated by the two analogs of fixed-sample-size unbiased estimators, namely by $U_{N_j}^{(1)} \equiv \left(\frac{N_j+1}{N_j}\right) X_{N_j:N_j}$ and $U_{N_j}^{(2)} \equiv 2\overline{X}_{N_j}$, for $j = 1, 2, 3, 4$.

The losses due to estimation of θ will be determined by the functions given in (1.5) and (1.6). Along the line of Robbins (1959), we define the usual "risk-efficiency" and "regret" of a sequential estimation procedure as:

$$\begin{array}{ll} \text{risk-efficiency:} & \eta_{N_j}^{(i)}(c) \equiv R_{N_j}^{(i)}(c;\theta)/R_{n^*}(c;\theta) \\ \text{regret} & \omega_{N_j}^{(i)}(c) = R_{N_j}^{(i)}(c;\theta) - R_{n^*}(c;\theta) \end{array} \tag{2.5}$$

for $i = 1, 2$. It should be clear that $i = 1$ or 2 respectively corresponds to the finally proposed estimator $U_{N_j}^{(1)}$ or $U_{N_j}^{(2)}$ for θ under each fixed $j = 1, 2, 3, 4$.

2.1. Properties of the Sequential Strategies $\left(N_j, U_{N_j}^{(i)}\right)$

It should be noted that this research is ongoing at this point. Hence, all of the asymptotic properties associated with the sequential strategies $\left(N_j, U_{N_j}^{(i)}\right), i = 1, 2, j = 1, 2, 3, 4$ have not yet been found. That said, we report what we have found so far without giving their proofs. However, we can easily show that $N_4 \leq N_1 \leq N_2$ w.p.1.

First, we summarize some of *asymptotic first-order* results (Ghosh and Mukhopadhyay, 1981) for the sequential strategies $\left(N_j, U_{N_j}^{(i)}\right)$ described by (2.1), (2.2), and (2.4).

Theorem 2.1. *For the sequential strategies* $\left(N_j, U_{N_j}^{(i)}\right)$ *described by (2.1), (2.2),* and *(2.4),* $j = 1, 2, 4$, *we have as* $c \to 0$:

(*i*) $N_j/n^* \to 1$ *w.p.1 under true* θ;

(*ii*) $E_\theta(N_j/n^*) \to 1$;

(*iii*) *For every fixed* $\varepsilon \in (0,1), P_\theta\left\{N_j \le [\varepsilon n^*]\right\} = O_e(c^{m/2})$ *where* $[u]$ *stands for the largest integer* $\le u$;

(*iv*) $\eta_{N_j}^{(i)}(c) \equiv R_{N_j}^{(i)}(c;\theta)/R_{n^*}(c;\theta) \to 1$, *that is, these estimation strategies are asymtotically risk-efficient*;

where $n^* = \left(2A\theta^2/c\right)^{1/3}$ *was defined by (1.8).*

Next, we summarize some of *asymptotic second-order* results (Ghosh and Mukhopadhyay, 1981) for the sequential strategies $\left(N_3, U_{N_3}^{(i)}\right)$ described by (2.3).

Theorem 2.2. *For the sequential strategies* $\left(N_3, U_{N_3}^{(i)}\right)$ *described by (2.3),* $i = 1, 2$, *we have as* $c \to 0$:

(*i*) $n^{*-1/2}(N_3 - n^*) \xrightarrow{\mathcal{L}} N(0, \frac{4}{27})$ *under true* θ;

(*ii*) $E_\theta(N_3) = n^* + \frac{5}{18} + o(1)$ for $m \ge 1$;

(*iii*) $E_\theta\left\{(N_3 - E(N_3))^2\right\} = \frac{4}{27}n^* + o(n^*)$ for $m \ge 1$.

(*iv*) $E_\theta\left\{(N_3 - n^*)^2\right\} = \frac{4}{27}n^* + o(n^*)$ for $m \ge 1$.

(*v*) *For every fixed* $\varepsilon \in (0,1), P_\theta\left\{N_j \le [\varepsilon n^*]\right\} = O_e(c^{m/2})$ *where* $[u]$ *stands for the largest integer* $\le u$;

(*vi*) $\eta_{N_j}^{(i)}(c) \equiv R_{N_j}^{(i)}(c;\theta)/R_{n^*}(c;\theta) \to 1$, *that is, these estimation strategies are asymtotically risk-efficient*;

where $n^* = \left(2A\theta^2/c\right)^{1/3}$ *was defined by (1.8).*

3. Data Analyses: Simulations

In this section, we briefly describe some simulation studies carried out in order to compare the performances of the randomly stopped versions of the two estimators $U_{N_j}^{(1)}$ and $U_{N_j}^{(2)}$. Our study includes the behavior of all four stopping variables from (2.1)-(2.4). Throughout, we keep $\theta = 1$ and $A = 1$ fixed, thus leading to $c = 2/n^{*3}$. We have considered a wide range of choices of n^* values.

But, for brevity, we summarize our findings in the case when $n^* = 150$ and $m = 10$. We start with $m = 10$ and then implement the sequential stopping rules (2.1)-(2.4) until termination. Under each configuration, we independently replicate each procedure 5000 times under each sampling strategy.

Suppose that at the k^{th} replication, we observe the stopping variable $N_j = n_{jk}$, the corresponding randomly stopped estimate for θ as $U_{N_{jk}}^{(i)} = u_{n_{jk}}^{(i)}$ for $i = 1, 2; j = 1, 2, 3, 4$, and $k = 1(1)5000$. The loss in estimating θ corresponding to $U_{N_{jk}}^{(1)}$ will be $L_{n_{jk}}^{(1)}\left(u_{n_{jk}}^{(1)}, \theta\right) = \left(u_{n_{jk}}^{(1)} - 1\right)^2 + cn_{jk}$ and that corresponding to $U_{N_{jk}}^{(2)}$ will be $L_{n_{jk}}^{(2)}\left(u_{n_{jk}}^{(2)}, \theta\right) = \frac{3}{n_{jk}}\left(u_{n_{jk}}^{(2)} - 1\right)^2 + cn_{jk}$. We use the following notations in our tables:

$$\overline{L}_{n_j}^{(i)} = \frac{1}{5000}\sum_{k=1}^{5000} L_{n_{jk}}^{(1)}\left(u_{n_{jk}}^{(1)}, \theta\right) \text{ for } i = 1, 2$$

$$\overline{n}_j = \frac{1}{5000} \sum_{k=1}^{5000} n_{jk}$$
$$s(n_j) = \sqrt{\frac{1}{(5000-1)} \sum_{k=1}^{5000} (n_{jk} - \overline{n}_j)^2}$$
$$R = (3/2)cn^*$$
$$\omega_{ij} = \frac{1}{5000} \sum_{k=1}^{5000} \widehat{\omega}_{n_{jk}}^{(i)}(c), \text{ for } i = 1, 2$$
$$\eta_{ij} = \frac{1}{5000} \sum_{k=1}^{5000} \widehat{\eta}_{n_{jk}}^{(i)}(c), \text{ for } i = 1, 2$$

where $j = 1, 2, 3, 4$. Then, we repeated this process 10 times independently.

Table 3.1. Comparing average and standard deviations of sample sizes from stopping rules (2.1)-(2.4): $m = 10, n^* = 150$

$\overline{n}_1$	s(n_1)	$\overline{n}_2$	s(n_2)	$\overline{n}_3$	s(n_3)	$\overline{n}_4$	s(n_4)
149.704	0.614	150.512	0.734	150.327	4.720	149.205	0.799
149.692	0.647	150.502	0.758	150.179	4.752	149.195	0.819
149.712	0.605	150.528	0.711	150.415	4.720	149.213	0.778
149.702	0.614	150.516	0.726	150.206	4.783	149.204	0.794
149.713	0.619	150.530	0.726	150.354	4.695	149.223	0.806
149.724	0.611	150.536	0.724	150.196	4.779	149.226	0.790
149.709	0.588	150.527	0.699	150.253	4.668	149.203	0.772
149.706	0.615	150.519	0.725	150.346	4.703	149.213	0.783
149.710	0.593	150.517	0.713	150.305	4.802	149.208	0.767
149.727	0.593	150.535	0.707	150.283	4.851	149.219	0.780

In Table 3.2, for immediate references, we provide all 10 values of $\overline{n}_j - n^*, j = 1, 2, 3, 4$.

Table 3.2. Comparing $\overline{n}_j - n^*, j = 1, 2, 3, 4$ values from stopping rules (2.1)-(2.4): $m = 10, n^* = 150$

$\overline{n}_1 - n^*$		$\overline{n}_2 - n^*$		$\overline{n}_3 - n^*$		$\overline{n}_4 - n^*$	
−0.296	−0.273	0.512	0.535	0.327	0.284	−0.795	−0.781
−0.288	−0.294	0.528	0.519	0.415	0.346	−0.787	−0.787
−0.287	−0.276	0.530	0.536	0.354	0.196	−0.777	−0.774
−0.291	−0.298	0.527	0.516	0.253	0.206	−0.797	−0.796
−0.290	−0.308	0.517	0.502	0.305	0.179	−0.792	−0.805

For brevity, we restrict our attention to only these two Tables 3.1-3.2 and Figure 3.1. Here, $n^* = 150$. Table 3.1 shows the estimated means and standard deviations of stopping times under four different rules. Table 3.2 gives estimated second-order results for the stopping times.

The non-linear renewal theory from Woodroofe (1977,1982) as well as from Lai and Siegmund (1977,1979) is used to prove the asymptotic second-order properties of stopping time N_3 as laid out in Theorem 2.2. Note that we may expect $E_\theta(N_3 - n^*) \approx$

$\frac{5}{18} = 0.278$. Columns 5-6 in Table 3.2 empirically confirms this. Also, from Theorem 2.2 we may expect $\sqrt{V_\theta(N_3)} \approx \sqrt{\frac{4}{27}n^*} = 4.714$.

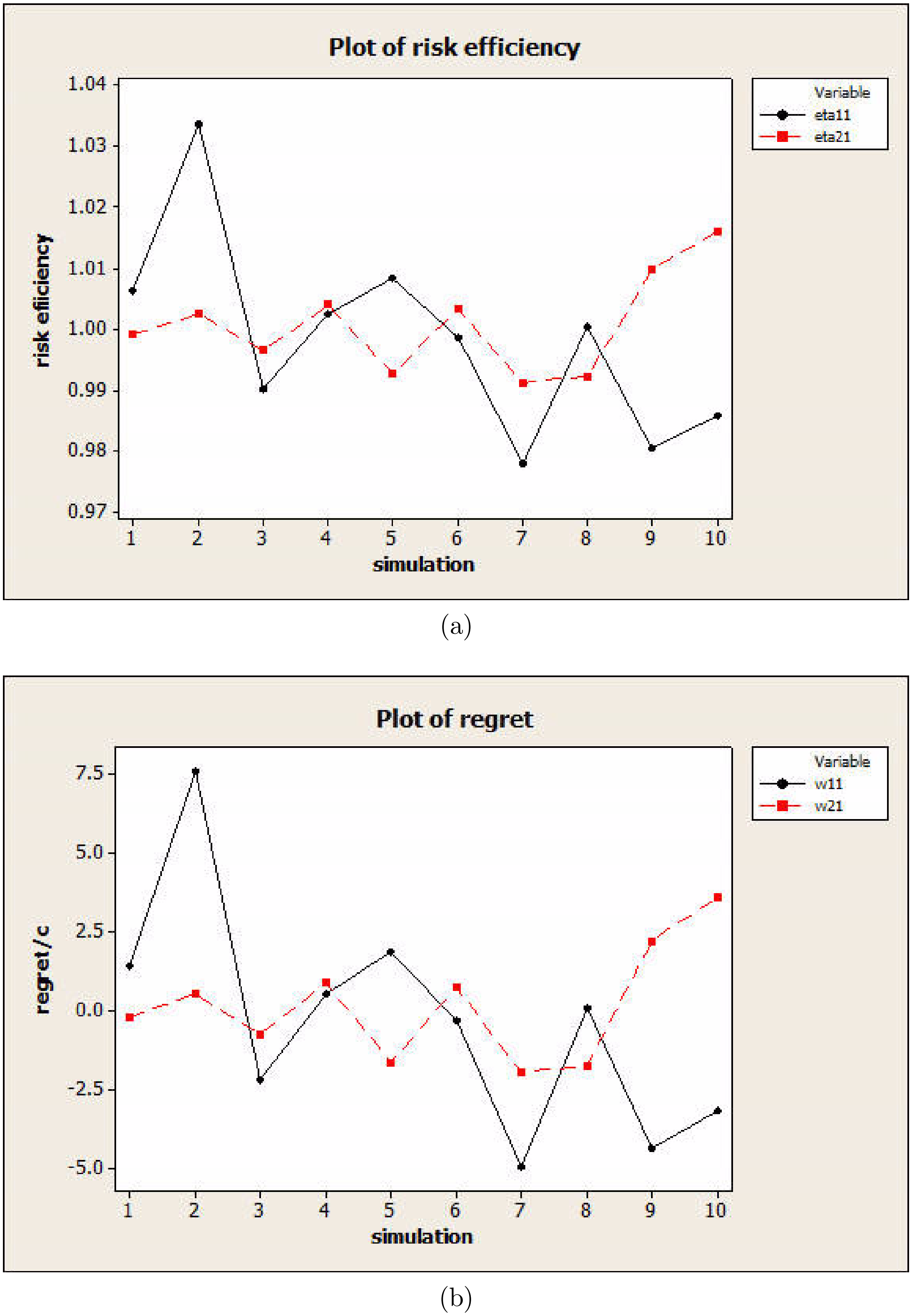

Figure 1: Plots of risk-efficiency (a) and regret (b) under stopping rule (2.1) when $m = 10$ and $n^* = 150$.

Column 6 in Table 3.1 empirically confirms this. Empirically, it appears that the estimated values of $\sqrt{V_\theta(N_j)} << $ the estimated values of $\sqrt{V_\theta(N_3)}, j = 1, 2, 4$.

However, for all four stopping rules, the average discrepancies between mean stopping time and n^* appears to be rather similar.

Figure 3.1 shows the behavior of risk and regret functions for 10 independent simulations corresponding to the stopping rule (2.1) only when $m = 10$ and $n^* = 150$. The randomly stopped estimator $2\overline{X}_{N_1}$ appears to perform better in comparison to the rival estimator $\left(\frac{N_1+1}{N_1}\right) X_{N_1:N_1}$. For brevity, we kept out other similar figures.

4. Concluding Thoughts

The focus of this article is to compare the behaviors of rival sequential versions of specific two specific fixed-sample-size unbiased estimators of θ, obtained from four different stopping rules. In the conference we hope to present more results and data analyses. We intend to show more details on how the regret and risk-efficiency behave for the two rival estimators under their respective loss functions.

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Abstract

The purely sequential minimum risk point estimation procedure for the unknown parameter θ (> 0) in a $U(0,\theta)$ population has been discussed in this article. This was developed under a squared error loss plus a linear cost function of sampling. The unknown parameter θ is estimated by means of four different estimators in the stopping rule, where as in the loss function two different unbiased estimators of θ were proposed. However, the unbiased estimators are randomly stopped versions of $X_{n:n}$ and $2\overline{X}_n$ in either loss function. Performances of such estimators are compared. Clearly, using a randomly stopped version of $2\overline{X}_n$ would amount to some loss of information when compared with a corresponding randomly stopped largest sample order statistic under both loss functions and the stopping rules. In this paper, we explore a novel approach to recover any loss of information by fine-tuning the loss function and then properly tailoring the associated sequential methodologies. We examine how the sequential risks of our newly proposed methodologies would compare with those associated with the existing sequential estimators.