Ambit fields and turbulence

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ABSTRACT

We discuss some potential applications of ambit processes and ambit fields to the modelling of the spatio–temporal dynamics of turbulent flows. Of key interest are the statistical properties of the turbulent velocity field.

1. INTRODUCTION

Stochastic modelling of the turbulent velocity field, understood as an explicit stochastic approach (in contrast to an implicit set–up in terms of governing equations and/or in terms of related quantities like velocity increments or velocity derivatives) is not well developed. Most of the existing literature on stochastic turbulence modelling deals with models for derived quantities like velocity increments, energy dissipation and accelerations. In addition, these models are mostly given in terms of (partial) stochastic differential equations that do not allow an explicit characterization of the solution.

Early attempts of modelling the turbulent velocity field go back to Kraichnan [33]. Kraichnan defines a Gaussian ensemble of velocities, decorrelated in time, which despite its strong limitations is still used in turbulent transport problems [24]. Gaussian velocity fields are also used in wind modelling with applications to wind energy farms [38]. Recently, this Gaussian model has been extended to account for non-Gaussian statistics by incorporating a multifractal model for the energy transfer from large to small scales to generate more realistic velocity increments [13].

In a Lagrangian framework, stochastic modelling of turbulence mainly concerns the modelling of velocity gradients or accelerations. Most of the stochastic Lagrangian models for the simulation of turbulent flows involve specific formulations of the generalized Langevin model [46]. The paper [15] discusses the Lagrangian time evolution of the full velocity gradient tensor and shows the potential of this model to reproduce many statistical and geometric trends observed in numerical and experimental three dimensional turbulent flows. This model is extended in [11] to a multiscale model that couples restricted Euler dynamics to a cascade model allowing energy exchange between scales to account for high Reynolds number dynamics. The question of universality of the intermittency based on the work in [15] is discussed in [16].

Some Lagrangian approaches also address the turbulent velocity field itself. In particular, boundary layer problems are discussed in [18, 26]. Second order stochastic models are considered in [10, 35, 47]. These papers include the modelling of the turbulent acceleration, which is also discussed in this context by conditioning on velocity fluctuations in [1]. Langevin models also proved useful in subgrid modelling as was shown in [19, 36, 37].

Another approach for modelling turbulent velocity increments has been initiated in [21, 22, 40] which investigates the Markov properties of turbulent velocity increments and accesses the multi-point probability densities of velocity increments by a Fokker-Planck equation. The same procedure has also been applied to the statistics of the energy dissipation in [25, 39, 42].

There is a huge amount of literature on stochastic modelling of various quantities in turbulent flows and the above discussion is definitely not covering all of it. But it gives an impression of the different approaches that are used and these approaches are, by nature, very different from the ambit field approach which we discuss in this paper.

Ambit fields allow to describe the turbulent velocity field explicitely as a function of time [5, 7]. The above mentioned models, in contrast, do not describe the velocity field itself and/or characterize the quantity of interest in terms of heuristic construction rules or (partial) stochastic differential equations that can not be solved in wide generality. In particular, we did not discuss the literature about the theory of the stochastic Navier Stokes equations (see [12, 20] and references therein) which mostly deals with existence and uniqueness properties of the solutions and does not capture the turbulent character beyond Kolmogorov scaling. In contrast, ambit fields provide a unifying stochastic modelling framework for all dynamically active scales in turbulent flows. Moreover, depending on the quantity of interest and the problem at hand, different approaches have been employed in the literature while the ambit field approach opens the possibility to incorporate all information about the turbulent velocity field and is at the same time flexible and mathematically tractable.

The organization of the paper is as follows. In Section 1.1 we give a brief account on the phenomenology of turbulent flows with focus on Kolmogorov's fundamental contributions. Ambit fields are defined in Section 1.2. They provide a general modelling framework for the turbulent velocity field. An important ingredient in this approach is the intermittency field which is modelled as a certain type of ambit field that is introduced in Section 1.3. A first application of the ambit framework to the modelling of turbulent time series is briefly discussed in Section 2. A possible spatio–temporal extension, the main topic of the present paper, is presented in Section 3. From these more general considerations we derive a simple toy–model in Section 4 that shows the essential properties of the new approach. Section 5 concludes with an outlook to possible extensions of the model.

1.1 Turbulence

There is no generally accepted definition of what should be called a turbulent flow. The main features of turbulent flows are low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time. Flow that is not turbulent is called laminar flow. The non-dimensional Reynolds number R determines whether flow conditions lead to a laminar or a turbulent state. Increasing the Reynolds number increases the turbulent character and the limit of infinite Reynolds number is called the fully developed turbulent state. In this paper, we will only consider turbulent flows that are stationary, homogeneous and isotropic.

Since the pioneering work of Kolmogorov [30, 31, 32] and Obukhov [43, 44, 45], intermittency of the turbulent velocity field plays a central role in turbulence research. Intermittency refers to the fact that fluctuations around the mean velocity occur in clusters and are more violent than expected from Gaussian statistics. In terms of moments of spatial velocity increments $v_t(x) - v_t(0)$, intermittency in turbulence is usually described by approximate multifractal scaling of spatial structure functions

$$(1) \quad \mathrm{S}_n(x) = \mathrm{E}\{(v_t(x) - v_t(0))^n\} \propto |x|^{\tau(n)}.$$

Here, $v_t(x)$ is one component of the velocity (usually along the mean flow) at position x (in direction of the mean flow) and time t and the lag x (in direction of the mean flow) is within the so-called inertial range. The inertial range is defined as the range of scales where the spectrum $E(k)$ (the Fourier transform of the correlation function of the velocity field) displays a power law $E(k) \propto k^{-5/3}$ [23, 43, 44]. The term multifractal scaling is due to the non-linear dependence of the spatial scaling exponents $\tau(n) > 0$ on the order n.

Scaling of second order and third order spatial structure functions can be motivated from the Navier–Stokes equation as was first exposed in Kolmogorov's seminal papers from 1941 [30, 31]. These scaling relation are called the 2/3rd law since $\tau(2) = 2/3$ and the 4/5th law since, for $x > 0$,

$$S_3(x) = -\frac{4}{5}\epsilon x$$

where ϵ is the mean energy dissipation. The energy dissipation measures the loss of kinetic energy in a turbulent flow due to friction forces.

Scaling of structure functions is usually measured in the time domain, invoking Taylor's Frozen Flow hypothesis [55]. In the time domain one observes approximate scaling of temporal structure functions

$$(2) \quad \hat{\mathrm{S}}_n(t) = \mathrm{E}\{(v_t(x) - v_0(x))^n\} \propto |t|^{\xi(n)}.$$

within a certain range of time scales t (the temporal counterpart to the inertial range). It is generally expected that $\tau(n) \approx \xi(n)$. (However, in [48] it is shown that $\tau(n) = \xi(n)$ can not hold exactly if Taylor's Frozen Flow hypothesis is exact.) In the time domain, third order structure functions $\hat{S}_3(s)$ show a positive skewness of temporal velocity increments for $s > 0$ within the temporal inertial range.

Multifractal scaling of velocity increments is assumed to hold exactly in the limit of infinite Reynolds numbers. However, experiments show that the scaling behaviour (2) might be poor, even for large Reynolds numbers [2, 54]. Furthermore, even if the scaling relation (2) holds, the inertial range still covers only part of the accessible scales where intermittency is observed.

From a probabilistic point of view intermittency refers, in particular, to the increase of the non-Gaussian behaviour of the probability density function (pdf) of velocity increments with decreasing scale. A typical scenario is characterized by an approximately Gaussian shape for the large scales (larger than scales at the inertial range), turning to exponential tails within the inertial range and stretched exponential tails for dissipation scales (below the inertial range) [14, 56]. It was reported in [4, 8] that the evolution of the pdf of velocity increments for all amplitudes and all scales can well be described within one class of tractable distributions, the normal inverse Gaussian (NIG) distributions (a list of references concerning NIG distributions is given in [4]).

There are many more stylized statistical features of turbulent flows and a proper stochastic model should be able to reproduce all of them. Of course, this is a far–reaching goal and a first step is to capture the most important stylized facts. In the present paper we will focus on scaling of spatial and temporal second order structure functions, a negative skewness of spatial third order structure functions, a positive skewness of temporal third order structure functions and the evolution of densities of velocity increments across scales within the class of NIG distributions. These properties set the minimal requirements we pose on our stochastic modelling framework.

1.2 Ambit fields

The concept of ambit fields arose out of a current study [5, 7, 49, 51, 52] the ultimate aim of which is to build a realistic stochastic process model of 3-dimensional turbulent velocity fields, in the spirit of Kolmogorov's phenomenological theory [23] – and beyond. Besides applications to turbulence, the concept has also been used in modelling the growth of cancer tumours [6, 27, 28, 50], in modelling electricity forward markets [3], and it should be of interest to other fields as well.

Let t denote time and x a point in some space $\mathcal{S}$. To each point $(x,t) \in \mathcal{S} \times \mathbb{R}$ let there be associated a random variable $Y_t(x)$. Such a process is said to be an *ambit field* if $Y_t(x)$ is determined as a weighted combination of past events in space-time coupled with innovations and where only events in a subset $A_t(x)$ of past space-time, termed an *ambit set*, is involved in determining $Y_t(x)$. The past events, except for the past innovations, are embodied in a random field $\{I_t(x)\}_{(x,t)\in\mathcal{S}\times\mathbb{R}}$.

Of special interest is the case of a *homogeneous and causal* family of ambit sets where

$$(3) \quad A_t(x) = \{(s,\xi) : (s-t, \xi - x) \in A_0(0)\}$$

and $A_0(0)$ is bounded by a function $\hat{h}$

$$(4) \quad A_0(0) = \left\{(s,\xi) : -\infty \le T \le s \le 0, |\xi| \le \hat{h}(|s|)\right\}$$

and where the boundary function $\hat{h}(s)$ is increasing on $[0,T]$. Here T denotes the temporal extension of the ambit set $A_t(x)$.

In the present paper we focus on ambit fields of the form

$$(5)\quad Y_t(x) = \int_{A_t(x)} f(t-s, x-\xi)\, L(\mathrm{d}s\mathrm{d}\xi)$$

where f is a damping function and L is a Lévy basis on $\mathbb{R}\times\mathcal{S}$, i.e. an independently and homogeneously scattered random measure whose values are infinitely divisible. For a mathematically more rigorous definition of independently scattered random measures and their theory of integration, see [5, 29, 34].

Note that $Y_t(x)$ is a stationary process in t for fixed x and a homogeneous process in x for fixed t.

1.3 Stochastic intermittency fields

We call the exponential of an ambit field a *stochastic intermittency field.* For ambit fields of the special type (5) we get

$$(6)\quad \sigma_t(x) = \exp\{Y_t(x)\} = \exp\left\{\int_{A_t(x)} f(t-s, x-\xi)\, L(\mathrm{d}s\mathrm{d}\xi)\right\}$$

These fields comprise, as special cases, the temporal cascade processes discussed in [41] and [53], as well as the spatio-temporal cascade process derived in [17, 52].

Independently scattered random measures provide a natural basis for describing uncorrelated noise processes in space and time. A special class is that of a homogeneous Lévy bases, where the distribution of the measure of each set is infinitely divisible and does not depend on the location of the set. In this case, it is easy to handle integrals with respect to the Lévy basis using the well-known Lévy-Khintchine and Lévy-Ito representations for Lévy bases. Here, we state the result and point to [5] for greater detail and rigour.

We have the fundamental relation

$$(7)\quad \mathrm{E}\left\{\exp\left\{\int_A f(a)\mathrm{L}(\mathrm{d}a)\right\}\right\} = \exp\left\{\int_A \mathrm{K}[f(a)]\mathrm{d}a\right\}$$

where E{ } denotes the expectation and K$\mathrm{d}a$ denotes the cumulant function of $L(\mathrm{d}a)$, defined by

$$(8)\quad \ln \mathrm{E}\left\{\exp\left\{\xi L(\mathrm{d}a)\right\}\right\} = \mathrm{K}[\xi]\mathrm{d}a.$$

The usefulness of (7) is obvious: it permits explicit calculation of the correlation function of the integrated and f-weighted noise field $L(\mathrm{d}a)$ once the cumulant function K is known.

The model (6) yields explicit expressions for arbitrary n-point correlations $\mathrm{E}\left\{\sigma_{t_1}(x_1)\cdot\ldots\cdot\sigma_{t_n}(x_n)\right\}$ in closed form. Here, the focus is on two-point correlators of order (n_1, n_2), defined as

$$(9)\quad c_{n_1,n_2}(x_1,t_1;x_2,t_2) \equiv \frac{\mathrm{E}\left\{\sigma_{t_1}(x_1)^{n_1}\sigma_{t_2}(x_2)^{n_2}\right\}}{\mathrm{E}\left\{\sigma_{t_1}(x_1)^{n_1}\right\}\mathrm{E}\left\{\epsilon(x_2,t_2)^{n_2}\right\}}.$$

Using (7), it is straightforward to show [5, 51, 52]

$$(10)\quad c_{n_1,n_2}(x_1,t_1;x_2,t_2) = \exp\left\{\int_{A_{t_1}(x_1)\cap A_{t_2}(x_2)} \overline{\mathrm{K}}[n_1 f(t_1-t, x_1-x), n_2 f(t_2-t, x_2-x)]\mathrm{d}x\mathrm{d}t\right\},$$

with the abbreviation $\overline{\mathrm{K}}[\lambda_1,\lambda_2] = \mathrm{K}[\lambda_1+\lambda_2] - \mathrm{K}[\lambda_1] - \mathrm{K}[\lambda_2]$.

In case of a finite ambit set with a finite decorrelation time T

$$(11)\quad A_0(0) = \left\{(s,\xi) : -T \le s \le 0, |\xi| \le \hat{h}(|s|)\right\}$$

and a constant weight function $f(t,x) = 1$ we get

$$(12)\quad c_{n_1,n_2}(x_1,t_1;x_2,t_2) = \exp\left\{ \overline{\mathrm{K}}[n_1,n_2] \int_{A_{t_1}(x_1)\cap A_{t_2}(x_2)} \mathrm{d}x\mathrm{d}t \right\},$$

The important point here is the fact that the exponent in (12) factorizes into the overlap of the two ambit sets times a factor depending only on the order (n_1, n_2).

2. MODELING TURBULENT TIME SERIES

High quality turbulent data sets, with a sufficient resolution of all dynamically active scales, mostly consist of measurements of the main component of the velocity vector at a fixed position in space and as a function of time. It has been shown in [7, 9, 49, 51, 52, 57] that *Brownian semistationary processes* (BSS) of the form

$$(13)\quad Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s + \beta \int_{-\infty}^{t} g(t-s)\sigma_s^2 \mathrm{d}s$$

constitute a suitable modelling framework for turbulent time series. Here, B is Brownian motion, g is a nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = 0$ for $t \leq 0$, and σ is a stationary càdlàg processes. The constant β controls the skewness of velocity increments.

BSS processes are able to reproduce many stylized features of turbulent time series, including scaling of structure functions, multiplier distributions, scaling of energy dissipation correlators, the evolution of the densities of velocity increments across scales and the statistics of the Kolmogorov variable. In this modelling applications σ_s is a stochastic intermittency field that can be interpreted as a continuous cascade process.

3. A SPATIO–TEMPORAL MODELING FRAMEWORK

A possible generalization of BSS processes to a spatio–temporal modelling framework for the main component of the velocity vector in one spatial dimension is the class of *spatially coupled Brownian semi–stationary fields* (SCBSS fields) defined as

$$(14)\quad Y_t(x) = \int_{-\infty}^{t} g(t-s)\sigma_s(x - c(t-s))\mathrm{d}B_s + \beta \int_{-\infty}^{t} g(t-s)\sigma_s^2(x - c(t-s))\mathrm{d}s$$

where σ is a stochastic intermittency field in space and time. The spatial dependence structure of $Y_t(x)$ is solely mediated by the intermittency field $\sigma_t(x)$.

In the turbulence context, the physical picture behind such a process is that of a constant *sweeping or advection velocity* c that transports the intermittent events $\sigma_t(x)$. The intermittency $\sigma_s(x - c(t-s))$ at position $x - c(t-s)$ reaches the point x at time t. Its contribution to $Y_t(x)$ is damped by $g(t-s)$ and modulated by $\mathrm{d}B_s$. The advection velocity is assumed to be positive, $c > 0$, i.e. the advection has the direction along the positive x–axis, which is also the direction of the mean velocity for $\beta > 0$.

We can think of σ^2 as some type of local kinetic energy input that accelerates the velocity field if $\mathrm{d}B_s > 0$ and slows it down if $\mathrm{d}B_s < 0$. The slowly varying skewness term then sums all the energy input that can reach the position x at time t. We thus conceive the intermittency field as a background field that sweeps with velocity c through the flow and leaves its fingerprint on the velocity at (x,t) damped by g and modulated by $\mathrm{d}B_s$.

3.1 Spatial structure functions of order two

Our primary goal is to suitably choose the ingredients of the spatio–temporal model (14) such that second order structure functions show a pronounced scaling behaviour within some inertial range. For the moment, we will ignore the skewness of the model (14), i.e. we set $\beta = 0$. We will discuss the case of a positive, but small β in Section 4.

In the spatial domain we want to model

$$(15)\quad S_2(x) = \mathrm{E}\left\{(Y_0(x) - Y_0(0))^2\right\} = Ax^{\tau(2)}$$

for $0 < 2x_0 \leq x \leq 2x_1$ and where A is some constant, i.e. for some inertial range bounded by the constant length scales $2x_0$ and $2x_1$.

We define

$$(16)\quad G(t) = \int_0^\infty g(s)g(s+t)\mathrm{d}s$$

and normalize the intermittency field σ such that $\mathrm{E}\{\sigma\} = 1$. We then get from the model (14) with $\beta = 0$

$$(17)\quad S_2(x) = 2G(0)\left(c_{1,1}(0,0;0,0) - c_{1,1}(x,0;0,0)\right)$$

where $c_{1,1}$ denotes the correlator (9) of order $(1,1)$. We thus have the necessary and sufficient condition for spatial inertial range scaling

$$(18)\quad c_{1,1}(0,0;0,0) - c_{1,1}(x,0;0,0) \propto x^{\tau(2)}$$

for $2x_0 \leq x \leq 2x_1$. To achieve this condition, we model the intermittency field σ as a stochastic intermittency field of the form

$$(19)\quad \sigma_t(x) = \exp\left\{\int_{x-\frac{L}{2}}^{x+\frac{L}{2}} \int_{t-T}^{t-T+h(|\rho-x|)} W(\mathrm{d}s\mathrm{d}\rho)\right\}.$$

Here L denotes a finite decorrelation–length and T denotes a finite decorrelation–time. The ambit set is bounded by the deterministic function h and the Lévy basis W is chosen to be a homogeneous normal Lévy basis on $\mathbb{R}^2$ with $W(\mathrm{d}a) \sim N(\mu\mathrm{d}a, \hat{\sigma}^2\mathrm{d}a)$. We choose

$$(20)\quad \mu = -\frac{\hat{\sigma}^2}{2}$$

and get the normalization $K[1] = 0$, i.e. $\mathrm{E}\{\sigma\} = 1$.

The spatial correlator of σ of order $(1,1)$ is, using (12)

$$(21)\quad c_{1,1}(x,0;0,0) = \exp\left\{2\overline{K}[1,1]\int_{\frac{x}{2}}^{\frac{L}{2}} \int_0^{h(\rho)} \mathrm{d}s\mathrm{d}\rho\right\}.$$

Here we assumed that h is decreasing. This condition is equivalent to the case of $\hat{h}$ increasing as defined in (4). The difference between the two bounding functions $\hat{h}$ and h is that the former is defined as a function of t and the latter as a function of x.

The next step is to find a suitable bounding function h such that the necessary condition (18) holds. For that we define

$$(22)\quad h(x) = \begin{cases} h_0(x), & 0 \leq x \leq x_0 \\ h_1(x), & x_0 \leq x \leq x_1 \\ h_2(x), & x_1 \leq x \leq \frac{L}{2} \end{cases}$$

and choose

(23) $h_1(x) = \dfrac{\tau(2) a_1 x^{\tau(2)-1}}{2\overline{K}[1,1]\left(a_0 - a_1 x^{\tau(2)}\right)}$

where a_0 and a_1 are positive constants and

(24) $a_0 > a_1 x_1^{\tau(2)}$

which ensures $h_1(x) > 0$.

We also need the condition $\frac{\partial h_1(x)}{\partial x} \leq 0$ which gives the condition

(25) $a_0 \geq \dfrac{a_1 x_1^{\tau(2)}}{1 - \tau(2)}.$

The condition (25) also implies condition (24).

Next, we calculate the spatial correlators of σ of order $(1,1)$. For that we define

$$
\begin{aligned}
A_0 &= \int_0^{x_0} h_0(\rho)\mathrm{d}\rho, \quad B_0 = \exp\left\{2\overline{K}[1,1]A_0\right\} \\
A_1 &= \int_{x_0}^{x_1} h_1(\rho)\mathrm{d}\rho, \quad B_1 = \exp\left\{2\overline{K}[1,1]A_1\right\} \\
A_2 &= \int_{x_1}^{\frac{L}{2}} h_2(\rho)\mathrm{d}\rho, \quad B_2 = \exp\left\{2\overline{K}[1,1]A_2\right\}
\end{aligned}
$$

and get for $2x_0 \leq x \leq 2x_1$

(26) $c_{1,1}(x,0;0,0) = \dfrac{B_2\left(a_0 - a_1 2^{-\tau(2)} x^{\tau(2)}\right)}{a_0 - a_1 x_1^{\tau(2)}}.$

Combining this result with

(27) $c_{1,1}(0,0;0,0) = B_0 B_1 B_2$

yields

(28) $c_{1,1}(0,0;0,0) - c_{1,1}(x,0;0,0) = \dfrac{B_2}{a_0 - a_1 x_1^{\tau(2)}}\left(B_0\left(a_0 - a_1 x_0^{\tau(2)}\right) - a_0 + a_1 2^{-\tau(2)} x^{\tau(2)}\right)$

and we get the spatial scaling relation

(29) $c_{1,1}(0,0;0,0) - c_{1,1}(x,0;0,0) = \dfrac{B_2 a_1 2^{-\tau(2)}}{a_0 - a_1 x_1^{\tau(2)}} x^{\tau(2)}$

if we have the second necessary condition (besides condition (25))

(30) $B_0\left(a_0 - a_1 x_0^{\tau(2)}\right) = a_0.$

This last condition can be rewritten as

(31) $\overline{K}[1,1] = \dfrac{\log(a_0) - \log(a_0 - a_1 x_0^{\tau(2)})}{2A_0}.$

In summary, we can construct the intermittency field σ solely from the condition of scaling of spatial structure functions of the velocity field under the assumption that $\beta = 0$. In this construction we can choose the parameters of the model for σ freely as long as the conditions (25) and (31) are fulfilled.

In addition, we are free to choose any small scale boundary $h_0(x)$ and any large scale boundary $h_2(x)$ as long as $h(x)$ is decreasing. This opens the possibility to include advanced modelling of spatial structure functions for scales below and above the inertial range.

3.2 Temporal structure functions of order two

Scaling of second order spatial structure functions (ignoring the skewness) fixes the inertial range ambit set of the intermittency field σ, but it gives no condition on the weight function g. As will be shown below, the weight function g is not a free parameter any more, if one additionally requires scaling of temporal structure functions of order two.

For the temporal structure functions we get, again assuming that $\beta = 0$,

$$(32)\quad \hat{S}_2(t) = \mathrm{E}\left\{(Y_t(0) - Y_0(0))^2\right\} = 2G(0)c_{1,1}(0,0;0,0) - 2G(t)c_{1,1}(ct,0;0,0).$$

A scaling relation

$$(33)\quad \hat{S}_2(t) = Bt^{\xi(2)}$$

for a constant B and for t within some temporal inertial range then yields the condition on g

$$(34)\quad G(t) = \frac{2G(0)c_{1,1}(0,0;0,0) - Bt^{\xi(2)}}{2c_{1,1}(ct,0;0,0)}$$

for t within the temporal inertial range. This condition on g involves the specification of $c_{1,1}(ct,0;0,0)$ for t within the temporal inertial range which is not restricted to coincide with the range $[2x_0/c, 2x_1/c]$. Of course, the correlator $c_{1,1}(ct,0;0,0)$ for t within the temporal inertial range must be chosen such that the condition (34) has a solution g. In the present paper we will not tackle these problems. The important point here is that it is possible to find a function g that produces scaling of temporal structure functions (or a good approximation to scaling) by proper choosing the orrelator $c_{1,1}(ct,0;0,0)$ for t within the temporal inertial range.

4. A SIMULATION STUDY

In this Section we illustrate the spatio–temporal turbulent velocity field (14) by simulations from a simple version of the model (14). Our purpose is not to determine the function g from the condition (34) but rather choose a simple weight–function g that shows the modelling potential of our stochastic framework concerning the evolution of densities of velocity increments across scales and the behaviour of the skewness of velocity increments.

For the model (14), we can freely choose $0 < \tau(2) < 1$, L, $x_1 < \frac{L}{2}$, a_1, $\overline{K}[1,1]$ and the constant a_0 according to (24). We then define the small scale boundary of the ambit set of σ as

$$(35)\quad h_0(x) = T$$

and the large scale boundary of the ambit set of σ as

$$(36)\quad h_2(x) = \frac{h_1(x_1)}{\frac{L}{2} - x_1}\left(\frac{L}{2} - x\right)$$

where, according to (31) and the condition that h must decay, we must have that

$$(37)\quad T = \frac{\log(a_0) - \log(a_0 - a_1 x_0^{\tau(2)})}{2\overline{K}[1,1]x_0} \geq h_1(x_0)$$

which gives a condition on x_0

$$(38)\quad \log\left(\frac{a_0}{a_0 - a_1 x_0^{\tau(2)}}\right) \geq \frac{\tau(2) a_1 x_0^{\tau(2)}}{a_0 - a_1 x_0^{\tau(2)}}$$

(there is always a possible interval for the choice of x_0).

Furthermore we choose, for simplicity,

$$(39)\quad g(t) = \begin{cases} \exp\{-\lambda t\} - \exp\{-\lambda T_0\}, & 0 \leq t \leq T_0 \\ 0, & \text{otherwise} \end{cases}$$

with positive constants λ and T_0. This function g does not fulfil condition (34) and we will not capture scaling of temporal structure functions in the simulations.

For the simulation of the model (14) we use the parameters listed in Table 1 and we choose (see condition (25))

$$(40)\quad a_0 = \frac{a_1 x_1^{\tau(2)}}{1-\tau(2)} + 1$$

We recorded a spatial resolution of stepsize $\Delta x = 1$ and a temporal resolution of stepsize $\Delta t = 1$.

Figure 1 shows the second order spatial structure function $S_2(x)$ in double logarithic representation. The straight line indicates an (approximate) scaling regime for the spatial inertial range $[11, 28]$ with the scaling exponent $\tau(2) = \frac{2}{3}$ in agreement with Kolmogorov's 2/3rd law. For $\beta = 0$ one would expect a spatial inertial range $[2x_0, 2x_1] = [2, 28]$. It is the positive skewness control parameter β that shrinks the spatial inertial range from below.

Figure 2 and Figure 3 display the third order spatial structure function $S_3(x)$ and the third order temporal structure function $\hat{S}_3(t)$, respectively. In the spatial domain we get a negative sign away from the very small scales and a positive sign in the temporal domain. This behaviour agrees with the empirical findings described in Section 1.1.

Some examples of the densities of the temporal velocity increments are shown in Figure 4 for a small, a medium and a large lag t. We observe the typical evolution across scales that is characteristic for turbulent time series, from heavy tales at small time lags towards an approximate Gaussian shape at large time lags. The solid lines show the approximation of the densities of velocity increments within the class of NIG distributions. The simulation produces distributions that are well fitted by NIG distributions and are very similar to empirical velocity increment distributions [4].

We further illustrate this similarity between simulated velocity increments and those from empirical turbulent time series within the NIG shape triangle in Figure 5. The NIG shape triangle displays the range of the steepness parameter ξ and the asymmetry parameter χ of NIG distributions

$$(41)\quad \{(\chi, \xi) : -1 < \chi < 1, 0 < \xi < \chi\}.$$

The normal distribution corresponds to the limiting point $(\chi, \xi) = (0, 0)$ and the Cauchy distribution to the limiting point $(\chi, \xi) = (0, 1)$. (We refer to [4] for a more detailed discussion of the NIG shape triangle and its relation to turbulence.) Figure 5 shows the location of the densities of temporal velocity increments from the simulation within the NIG shape triangle together with the location of the densities of velocity increments from a turbulent time series measured in the atmospheric boundary layer [4]. The evolution of the densities of velocity increments across scales from the simulation is very similar to the typical evolution observed for turbulent data sets.

5. SUMMARY AND OUTLOOK

The spatio–temporal model (14) is able to reproduce scaling of spatial and temporal structure functions of order two, a positive skewness for temporal velocity increments, a negative skewness for spatial velocity increments and an evolution of the densities of velocity increments across scales within the class of NIG distributions that is very similar to empirical findings. To show the distributional properties we used a very simple version of the model that does not have the condition (34). The determination of g from the condition (34) is the next step in a further refinement of the model.

The basic assumption underlying the spatio–temporal model (14) is that of a constant sweeping velocity c in direction of the positive x–axis. Keeping this assumption, a possible extension to a spatio–temporal model with 3 space dimensions is straightforward. One can also argue that the actual sweeping velocity is not a constant but bounded by some maximum value c_m. Taking these arguments into account and assuming that the advection is directional in direction of the positive x–axis, we arrive at the following extended spatio–temporal model in $3+1$ dimensions

$$(42)\quad Y_t(\vec{r}) = \int_{-\infty}^{t}\int_{0}^{c_m} g_c(t-s)\sigma_s(\vec{r}-c(t-s)\vec{e})\mathrm{W}(\mathrm{d}c\mathrm{d}s) + \beta\int_{-\infty}^{t}\int_{0}^{c_m} g_c(t-s)\sigma_s^2(\vec{r}-c(t-s)\vec{e})\mathrm{d}c\mathrm{d}s.$$

Here, $\vec{e}$ is the unit vector in x–direction, W is a homogeneous normal Lévy basis on $[0, c_m] \times \mathbb{R}$ and the damping function g is allowed to depend on the sweeping velocity c. The intermittency field σ is a stochastic intermittency field with the possibility to model anisotropic spatial structure functions by choosing a proper anisotropic ambit set.

A further generalization would be to include density fluctuations that travel undirectional with the speed of sound c_0 and constitute a second mechanism that connects the dynamics at different space–time points.

References

[1] Aringazin, A. K. and Mazhitov, M. I. (2004): One-dimensional Langevin models of fluid particle acceleration in developed turbulence. *Phys. Rev. E* **69**, 026395.

[2] Arneodo, A., Baudet, C., Belin, F., Benzi, R., Castaing, B., Chabaud, B., Chavarria, R., Ciliberto, S., Camussi, R., Chillá, F., Dubrulle, B., Gagne, Y., Hebral, B., Herweijer, J., Marchand, M., Maurer, J., Muzy, J. F., Naert, A., Noullez, A., Peinke, J., Roux, F., Tabeling, P., van de Water, W. and Willaime, H. (1996): Structure functions in turbulence, in various flow configurations, at Reynolds number between 30 and 5000, using extended self-similarity. *Europhys. Lett.* **34**, 411-416.

[3] Barndorff-Nielsen, O. E., Benth, F. E. and Veraart, A. E. D. (2010): Modelling electricity forward markets by ambit fields. Research Paper 2010-41. Creates, University of Aarhus.

[4] Barndorff-Nielsen, O. E., Blæsild, P. and Schmiegel, J. (2004): A parsimonious and universal description of turbulent velocity increments. *Eur. Phys. J. B* **41**, 345-363.

[5] Barndorff-Nielsen, O. E. and Schmiegel, J. (2004): Lévy based tempo-spatial modelling; with applications to turbulence. *Uspekhi Mat. Nauk* **159**, 63-90.

[6] Barndorff-Nielsen, O. E. and Schmiegel, J. (2007): Ambit processes; with applications to turbulence and cancer growth. In Benth, F.E., Nunno, G.D., Linstrøm, T., Øksendal, B. and Zhang, T. (Eds.): *Stochastic Analysis and Applications: The Abel Symposium 2005.* Heidelberg: Springer. Pp. 93-124.

[7] Barndorff-Nielsen, O. E. and Schmiegel, J. (2008a): A stochastic differential equation framework for the timewise dynamics of turbulent velocities. *Theory Prob. Its Appl.* **52** , 372-388.

[8] Barndorff-Nielsen, O. E. and Schmiegel, J. (2008b): Time change, volatility and turbulence. In Sarychev, A., Shiryaev, A., Guerra, M. and Grossinho, M.d.R. (Eds.): *Proceedings of the Workshop on Mathematical Control Theory and Finance.* Lisbon 2007. Berlin: Springer. Pp. 29-53.

[9] Barndorff-Nielsen, O.E. and Schmiegel, J. (2009): Brownian semistationary processes and volatility/intermittency. In Albrecher, H., Rungaldier, W. and Schachermeyer, W. (Eds.): *Advanced Financial Modelling*. Radon Series Comp. Appl. Math. **8**. Pp. 1-26.

[10] Beck, C. (2002): Lagrangian acceleration statistics in turbulent flows. *Europhys. Lett.* **64**, 151-157.

[11] Biferale, L., Chevillard, L., Meneveau, C. and Toschi, F. (2007): Multiscale model of gradient evolution in turbulent flows. *Phys. Rev. Lett.* **98**, 214501.

[12] Birnir, B. (2008): Turbulence of a unidirectional flow. In M. Pinsky and B. Birnir (Eds.): *Probability, Geometry and Integrable Systems*. Cambridge: Cambridge University Press.

[13] Biss, K. H. (2009): Windfeldmodellierung mit multifraktaler Turbulenz. Diploma thesis, Justus-Liebig University Giessen, Germany.

[14] Castaing, B., Gagne, Y. and Hopfinger, E. J. (1990): Velocity probability density functions of high Reynolds number turbulence. *Physica D* **46**, 177-200.

[15] Chevillard, L. and Meneveau, C. (2006): Lagrangian dynamics and statistical geometric structure of turbulence. *Phys. Rev. Lett.* **97**, 174501.

[16] Chevillard, L. and Meneveau, C. (2007): Intermittency and universality in a Lagrangian model of velocity gradients in three-dimensional turbulence. *C. R. Mecanique* **335**, 187-193.

[17] Cleve, J., Schmiegel, J. and Greiner, M. (2008): Apparent scale correlations in a random multifractal process. *Europ. Phys. J. B* **63**, 109-116.

[18] Dehbi, A. (2008): Turbulent particle dispersion in arbitrary wall-bounded geometries: a coupled CFD-Langevin-equation approach. *Int. J. Multiphase Flow* **34**, 819-828.

[19] Dubrulle, B., Laval, J. P., Mazarenko, S. and Kevlahan, N. K. R. (2001): A dynamic subfilter-scale model for plane parallel flows. *Phys. Fluids* **13**, 2045-2064.

[20] Flandoli, F. (2008): An Introduction to 3D Stochastic Fluid Dynamics. In G. Da Prato and M. Röckner (Eds.): *SPDE in Hydrodynamic*. "Lecture Notes in Mathematics" Series. Berlin: Springer. Pp. 51-150.

[21] Friedrich, R. and Peinke, J. (1997a): Description of a turbulent cascade by a Fokker-Planck equation. *Phys. Rev. Lett.* **78**, 863-866.

[22] Friedrich, R. and Peinke, J. (1997b): Statistical properties of a turbulent cascade. *Physica D* **102**, 147-155.

[23] Frisch, U. (1995): *Turbulence. The legacy of A.N. Kolmogorov.* Cambridge University Press, Cambridge.

[24] Gawędzki, K. (2008): Stochastic processes in turbulent transport. Preprint http://arxiv.org/abs/0806.1949.

[25] Hosokawa, I. (2002): Markov process built in scale-similar multifractal energy cascades in turbulence. *Phys. Rev. E* **65**, 027301.

[26] Iliopoulos, I., Mito, Y. and Hanratty, T. J. (2003): A stochastic model for solid particle dispersion in a nonhomogeneous turbulent field. *Int. J. Multiphase Flow* **29**, 375-394.

[27] Jensen, E. B. V., Jonsdottir, K. Y., Schmiegel, J. and Barndorff-Nielsen, O. E. (2005): Spatio-temporal modelling - with a view to biological growth. In Finkenstädt, B., Held, L. and Isham, V. (Eds.): *Statistics of Spatio-Temporal Systems*. Monographs on Statistics and Applied Probability. Chapman & Hall/CRC. Pp. 47-75.

[28] Jonsdottir, K. Y., Schmiegel, J. and Jensen, E. B. V. (2008): Lévy based growth models. *Bernoulli* **14**, 62-90.

[29] Kallenberg, O. (1989): *Random Measures*. Akademie Verlag, Berlin.

[30] Kolmogorov, A. N. (1941a): The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk. SSSR* **30**, 299-303.

[31] Kolmogorov, A. N. (1941b): Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk. SSSR* **32**, 16-18.

[32] Kolmogorov, A. N. (1962): A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds numbers. *J. Fluid Mech.* **13**, 82-85.

[33] Kraichnan, R. H. (1968): Small-scale structure of a scalar field convected by turbulence. *Phys. Fluids* **11**, 945-963.

[34] Kwapien, S. and Woyczynski, W. A. (1992): *Random Series and Stochastic Integrals: Single and Multiple.* Birkhäuser, Basel.

[35] Lamorgese, A. G., Pope, S. B., Yeung, P. K. and Sawford, B. L. (2007): A conditionally cubic-Gaussian stochastic Lagrangian model for acceleration in isotropic turbulence. *J. Fluid Mech.* **582**, 399-422.

[36] Laval, J. P., Dubrulle, B. and McWilliams, J. C. (2003): Langevin models of turbulence: renormalization group, distant interaction algorithms or rapid distortion theory? *Phys. Fluids* **15**, 1327-1339.

[37] Laval, J. P. and Dubrulle, B. (2006): A LES-Langevin model for turbulence. *Eur. J. Phys. B* **49**, 471-481.

[38] Mann, J. (1998): Wind field simulation. *Prob. Engng. Mech.* **13**, 269-282.

[39] Marcq, P. and Naert, A. (1998): A Langevin equation for the energy cascade in fully developed turbulence. *Physica D* **124**, 368-381.

[40] Marcq, P. and Naert, A. (2001): A Langevin equation for turbulent velocity increments. *Phys. Fluids* **13**, 2590-2595.

[41] Muzy, J. F. and Bacry, E. (2002): Multifractal stationary random measures and multifractal random walks with log infinitely divisible scaling laws. *Phys. Rev. E* **66**, 056121.

[42] Naert, A., Friedrich, R. and Peinke, J. (1997): Fokker-Planck equation for the energy cascade in turbulence. *Phys. Rev. E* **56**, 6719-6722.

[43] Obukhov, A. M. (1941a): On the distribution of energy in the spectrum of turbulent flow. *Dokl. Akad. Nauk SSSR* **32**(1), 22-24.

[44] Obukhov, A. M. (1941b): Spectral energy distribution in a turbulent flow. *Izv. Akad. Nauk SSSR Ser. Geogr. Geofiz* **5**(4-5), 453-466.

[45] Obukhov, A. M. (1962): Some specific features of atmospheric turbulence. *J. Fluid Mech.* **13**, 77-81.

[46] Pope, S. B. (2000): *Turbulent Flows.* Cambridge University Press, Cambridge.

[47] Reynolds, A. M. (2003): On the application of nonextensive statistics to Lagrangian turbulence. *Phys. Fluids* **15**, L1.

[48] Schmiegel, J. (2003): Some Remarks on Scaling and Universality in Turbulence. Research Report 2003-19. MaPhySto, University of Aarhus.

[49] Schmiegel, J. (2005): Self-scaling of turbulent energy dissipation correlators. *Phys. Lett. A* **337**, 342-353.

[50] Schmiegel, J. (2006): Self-scaling tumor growth. *Physica A* **367**, 509-524.

[51] Schmiegel, J., Barndorff-Nielsen, O. E. and Eggers, H. C. (2006): A Class of Spatio-Temporal and Causal Stochastic Processes, with Application to Multiscaling and Multifractality. *South African Journal of Science* **101**, 513-519.

[52] Schmiegel, J., Cleve, J., Eggers, H. C., Pearson, B. R. and Greiner, M. (2004): Stochastic energy-cascade model for (1+1)-dimensional fully developed turbulence. *Phys. Lett. A* **320**, 247-253.

[53] Schmitt, F. G. (2003): A causal multifractal stochastic equation and its statistical properties. *Eur. Phys. J. B* **34**, 85-98.

[54] Sreenivasan, K. R. and Antonia, R. A. (1997): The phenomenology of small–scale turbulence. *Ann. Rev. Fluid Mech.* **29**, 435-472.

[55] Taylor, G. I. (1938): The spectrum of turbulence. *Proc. R. Soc. Lond. A* **164**, 476-490.

[56] Vincent, A. and Meneguzzi, M. (1991): The spatial structure and statistical properties of homogeneous turbulence. *J. Fluid Mech.* **225**, 1-25.

[57] For a detailed case study of the modelling procedure, we refer to the contribution of Emil Hedevang at this conference.

λ	L	T_0	$\tau(2)$	a_1	$\overline{K}[1,1]$	x_0	x_1	c
0.1	30	30	2/3	1	0.001	1	14	1

Table 1: Simulation parameters

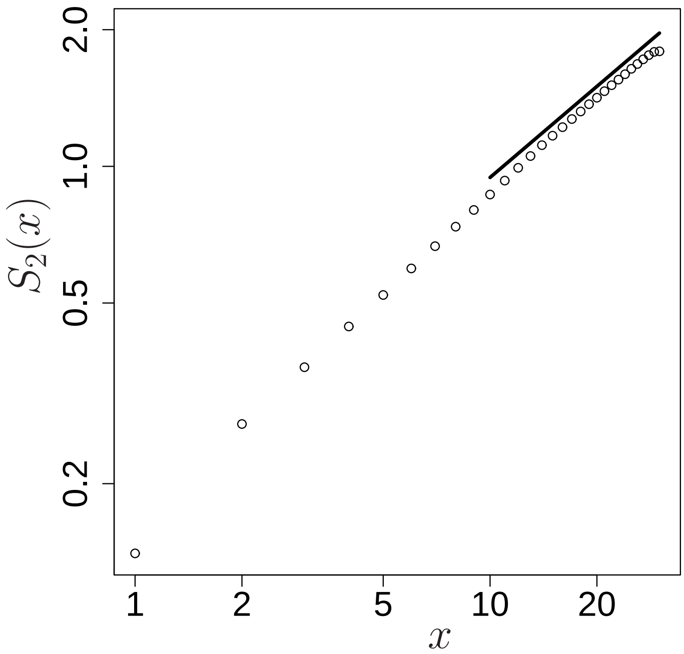

Figure 1: Double logarithmic representation of the second order spatial structure function $S_2(x) = \mathrm{E}\{(Y_0(x) - Y_0(0))^2\}$ obtained from a simulation of the model (14). The parameters for the simulation are described in Section 4. The solid line indicates the scaling behaviour $\propto x^{2/3}$.

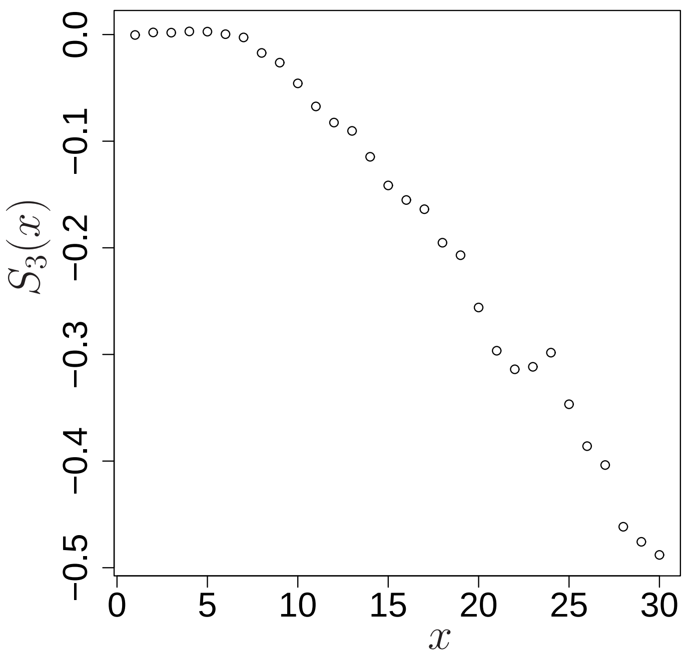

Figure 2: The third order spatial structure function $S_3(x) = \mathrm{E}\{(Y_0(x) - Y_0(0))^3\}$ obtained from a simulation of the model (14). The parameters for the simulation are described in Section 4.

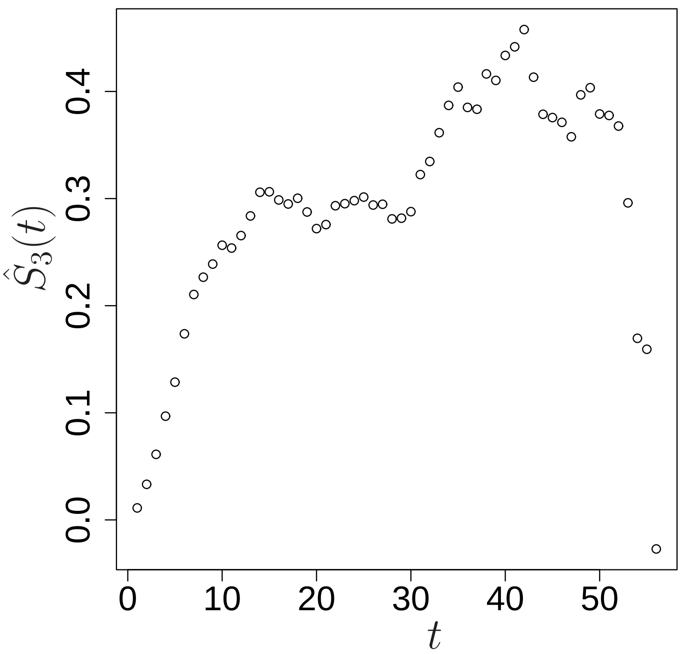

Figure 3: The third order temporal structure function $\hat{S}_3(t) = \mathrm{E}\{(Y_t(0) - Y_0(0))^3\}$ obtained from a simulation of the model (14). The parameters for the simulation are described in Section 4.

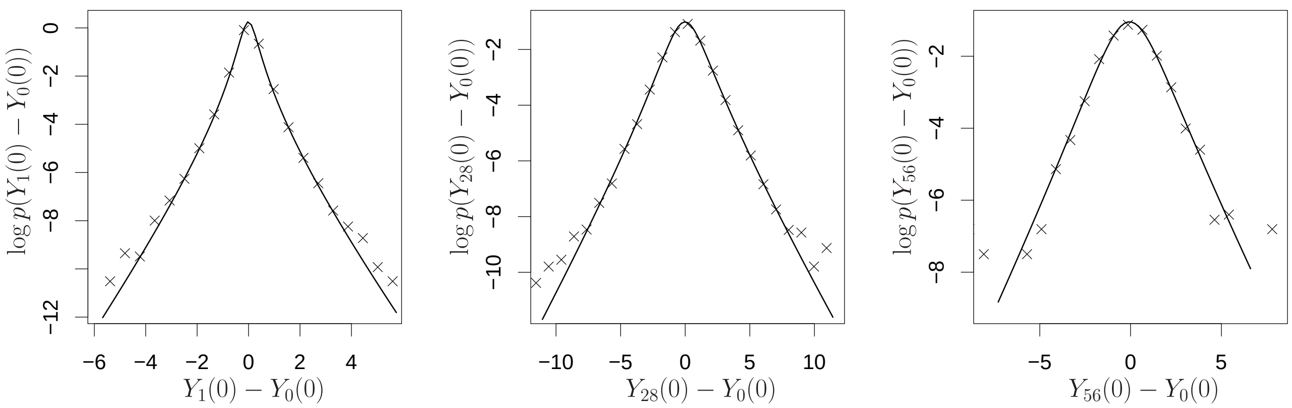

Figure 4: The log densities $\log p(Y_t(0) - Y_0(0))$ of temporal velocity increments $Y_t(0) - Y_0(0)$ for $t = 1, 28, 56$, obtained from a simulation of the model (14). The parameters for the simulation are decribed in Section 4.

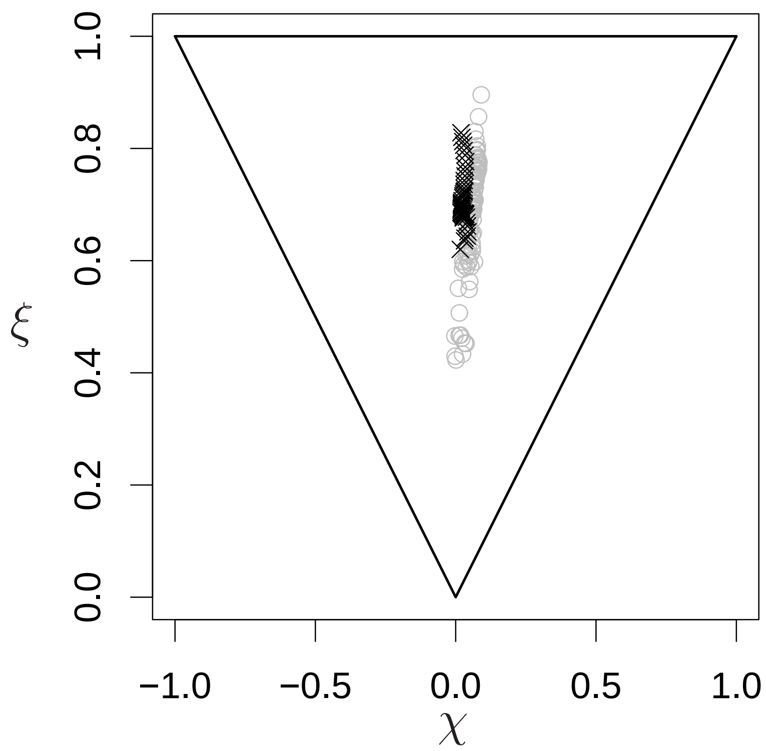

Figure 5: NIG shape triangle for the evolution of the densities of temporal velocity increments $Y_t(0) - Y_0(0)$ obtained from a simulation of the model (14) ($\times$) (black) with increasing t from top to bottom. The parameters for the simulation are described in Section 4. For comparison, we also show the evolution of the densities of temporal velocity increments across scales for a data set measured in the atmospheric boundary layer ($\circ$) (gray).